Eisenstein congruences in higher rank

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Eisenstein congruences for weight 2 modular forms

Theorem (Mazur 1977)

N - prime, $p|\frac{N-1}{24}$, $\mathfrak{p}|p$ in $\overline{\mathbb{Q}}$
$E_{2,N}$ - Eisenstein series of weight 2 and level N
There exists a newform $f \in S_2(N)$ such that

$$f \equiv E_{2,N} \mod p \quad (i.e. \mod \mathfrak{p}).$$

- Applications to non-vanishing Fourier coefficients, rational points on elliptic curves and Jacobians, L-values, ...
- Ribet (∼2010, unpub[1]) and [Yoo 2019] extended to squarefree N assuming $p \geq 5$, $p \nmid N$
- [M. 2017] extended to more general N, p, HMFs, ...

[1] Remember TORA IV (2013)!

A sample result for $\mathrm{U}(2m+1)$

Theorem (M.–Wakatsuki)

$n = 2m+1$ prime, $E/F = \mathbb{Q}(i)/\mathbb{Q}$, $G = \mathrm{U}_{E/F}(n)$ (quasi-split), π_0 - pseudo-Eisenstein series $\leftrightarrow \mathrm{Ind}_B^G(1)$, $p > n$, $\ell \equiv 1 \mod 4$ prime. Suppose

1. $p|(\ell^r - 1)$ *for some* $1 \leq r \leq n-1$*; or*
2. $p|\prod_{r=1}^m B_{2r} \prod_{r=1}^m B_{2r+1,\chi_{E/F}}$.

Then there exists a holomorphic weight n cuspidal π of $G(\mathbb{A})$ which is (i) spherical outside of ℓ, (ii) unramified twist of Steinberg at ℓ, (iii) non-endoscopic, and (iv) Hecke congruent to π_0 mod p.

- assumes endoscopic classification $(n > 3)$
- similar results for any CM E/F
- weaker results for more general groups

Bernoulli conditions

Case 2 of Theorem says:

n	num($\prod_{r=1}^{m} B_{2r} \prod_{r=1}^{m} B_{2r+1,\chi_{E/F}}$)
3	1
5	5
7	$5 \cdot 61$
11	$5^2 \cdot 19 \cdot 61 \cdot 277 \cdot 2659$
13	$5^2 \cdot 13 \cdot 19 \cdot 43 \cdot 61 \cdot 277 \cdot 691 \cdot 967 \cdot 2659$
17	$5^3 \cdot 13 \cdot 17 \cdot 19 \cdot 43 \cdot 47 \cdot 61 \cdot 277 \cdot 691 \cdot 967 \cdot 2659$ $\cdot 3617 \cdot 4241723 \cdot 228135437$
19	$5^3 \cdot 13 \cdot 17 \cdot 19 \cdot 43 \cdot 47 \cdot 61 \cdot 79 \cdot 277 \cdot 349 \cdot 691$ $\cdot 967 \cdot 2659 \cdot 3617 \cdot 43867 \cdot 4241723 \cdot 87224971$ $\cdot 228135437$

if $p|$ RH column, get Eisenstein congruence on $\mathrm{U}(n)$ of weight n and "Steinberg level ℓ" for any $\ell \equiv 1 \mod 4$

Eisenstein Hecke eigenvalues: split places

Basis of local Hecke operators for $\mathrm{GL}_n(\mathbb{Q}_q)$:

$$T_r = \mathrm{diag}(\underbrace{q, \dots, q}_{r}, \underbrace{1, \dots, 1}_{n-r}) \quad 1 \leq r < n.$$

T_r-eigenvalue for π_0 is

$$\lambda_r = q^{\frac{(n-r)r}{2}} e_r(q^{\frac{n-1}{2}}, q^{\frac{n-3}{2}}, \dots, q^{\frac{1-n}{2}}),$$

where e_r is r-th elementary symmetric polynomial

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any n: $\lambda_1 = q^{n-1} + q^{n-2} + \cdots + q + 1$

$n = 2$: $\lambda_1 = q + 1$

$n = 3$: $\lambda_1 = \lambda_2 = q^2 + q + 1$

$n = 5$: $\lambda_1 = q^4 + q^3 + q^2 + q + 1$

$\lambda_2 = \lambda_3 = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$

Eisenstein Hecke eigenvalues: unramified inert places

Basis of local Hecke operators for $\mathrm{U}_n(\mathbb{Q}_q)$, $n = 2m+1$:

$$T_r = \mathrm{diag}(\underbrace{q, \ldots, q}_{r}, \underbrace{1, \ldots, 1}_{n-2r}, \underbrace{q^{-1}, \ldots, q^{-1}}_{r}) \quad 1 \leq r < m.$$

T_r-eigenvalue for π_0 is

$$\lambda_r = q^{(2m-r+1)r} e_r(q^{2m} + q^{-2m}, q^{2(m-1)} + q^{-2(m-1)}, \ldots, q^2 + q^{-2}),$$

where e_r is r-th elementary symmetric polynomial

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any $n = 2m+1$: $\lambda_1 = q^{4m} + q^{4m-2} + \cdots + 1 - q^{2m}$

$n = 2$: $\lambda_1 = q^2 + 1$

$n = 3$: $\lambda_1 = q^4 + 1$

$n = 5$: $\lambda_1 = q^8 + q^6 + q^2 + 1$

$\lambda_2 = q^{12} + q^8 + q^4 + 1$

Outline of Method

F – totally real number field
G – reductive F-group such that G has a compact real inner form
e.g.: $\mathrm{U}(n)$, $\mathrm{PGL}(2)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$—not $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ $(n \geq 3)$

Goal: Construct Eisenstein congruences (wrt Hecke eigenvalues) on G.

1. Choose an inner form G^* of G such that $G^*(F_\infty)$ is compact.
2. Use mass formula to construct "Eisenstein" congruence on G^* for algebraic modular forms of trivial weight
3. Transfer congruence to G via functoriality (generalized Jacquet–Langlands correspondence)

Working example: Step 1

Step 1: Choose an inner form G^* of G such that $G^*(F_\infty)$ is compact.

Working example: $G = \mathrm{U}(3)$ (quasi-split) for $E/F = \mathbb{Q}(i)/\mathbb{Q}$

$$G = \left\{ g \in \mathrm{GL}_3(\mathbb{Q}(i)) : {}^t\overline{g}\Phi g = \Phi \right\}, \quad \Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

One inner form compact at infinity is:

$$G^* = \mathrm{U}(3)^* = \left\{ g \in \mathrm{GL}_3(\mathbb{Q}(i)) : {}^t\overline{g} g = I \right\}$$

Algebraic modular forms I

$G^* = \mathrm{U}(3)^* = \{g \in \mathrm{GL}_3(\mathbb{Q}(i)) : {}^t\overline{g}g = I\}$
Hermitian lattice $\Lambda = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus 5\mathbb{Z}[i]$
$K = \prod_{v<\infty} K_v \times G^*(\mathbb{R})$, $K_v = \mathrm{Stab}_{G_v^*}(\Lambda_v)$
$\mathrm{Cl}(K) = \mathrm{Cl}(\Lambda) = G^*(\mathbb{Q})\backslash G^*(\mathbb{A})/K$ — classes of genera of Λ
Write $\mathrm{Cl}(K) = \{x_1, x_2, \ldots, x_h\} \leftrightarrow \{\Lambda_1, \ldots, \Lambda_h\}$

Algebraic modular forms with trivial weight level K:

$$A(G^*, K) = \{\phi : \mathrm{Cl}(K) \to \mathbb{C}\}$$

Inner product on $A(G^*, K)$:

$$(\phi, \phi') = \sum \frac{1}{w_i}\phi(x_i)\overline{\phi'(x_i)}, \quad w_i = |\mathrm{Aut}(\Lambda_i)|.$$

$$A(G^*, K) = \bigoplus \pi^K, \quad \pi \text{ aut rep of } G^*(\mathbb{A})$$

Algebraic modular forms II

$$A(G^*, K) = \{\phi : \mathrm{Cl}(K) \to \mathbb{C}\}, \quad \mathrm{Cl}(K) = G^*(\mathbb{Q}) \backslash G^*(\mathbb{A}) / K$$

$$(\phi, \phi') = \sum \frac{1}{w_i} \phi(x_i) \overline{\phi'(x_i)}, \quad w_i = |\mathrm{Aut}(x_i)|.$$

$$A(G^*, K) = \bigoplus \pi^K, \quad \pi \text{ aut rep of } G^*(\mathbb{A})$$

$$A(G^*, K) = \mathbb{C}\mathbf{1} \oplus A_0(G^*, K), \quad A_0(G^*, K) = \{\phi : (\phi, \mathbf{1}) = 0\}$$

Have Hecke action...

Our example: $|\mathrm{Cl}(K)| = 3$. Table of eigenforms:

w_i	128	32	24
	$\phi(x_1)$	$\phi(x_2)$	$\phi(x_3)$
$\mathbf{1}$	1	1	1
ϕ_1	16	$-4(2+\sqrt{5})$	$3(1+\sqrt{5})$
ϕ_2	16	$-4(2-\sqrt{5})$	$3(1-\sqrt{5})$

Working example: Step 2

Step 2: Construct $\phi \in A_0(G^*, K)$ Hecke congruent to $\mathbf{1}$.

w_i	128	32	24
	$\phi(x_1)$	$\phi(x_2)$	$\phi(x_3)$
$\mathbf{1}$	1	1	1
ϕ_1	16	$-4(2+\sqrt{5})$	$3(1+\sqrt{5})$
ϕ_2	16	$-4(2-\sqrt{5})$	$3(1-\sqrt{5})$

mass $m(K) = \sum \frac{1}{w_i} = (\mathbf{1}, \mathbf{1}) = \frac{31}{384}$

- $p | m(K) \implies \exists \phi \in A_0(G^*, K)$ s.t. $\phi \equiv \mathbf{1} \mod p$
  e.g., $\phi(x_1) = \phi(x_2) = 32, \phi(x_3) = -30$ $(p = 31)$
- $\phi \mod p$ is a $\mathrm{mod}\ p$ Hecke eigenform; lifting argument (e.g., Deligne–Serre) $\rightsquigarrow$ Hecke eigenform $\phi' \in A_0(G^*, K)$ Hecke congruent to $\mathbf{1} \bmod p$
  (in fact $2\phi_1 \equiv 2\phi_2 \equiv \mathbf{1} \bmod 31$)

Step 3

Step 3: Transfer congruence on G^* to congruence on G

Issues (assuming endoscopic classification for G^*, G):
— transfer to G may not be cuspidal
— transfer to G may be endoscopic

- for $G = \mathrm{SO}(5) = \mathrm{PGSp}(4)$, construction gives Eisenstein congruences for weight 3 Siegel modular forms, but these congruences can (typically) be explained as Saito–Kurokawa lifts of weight 4 Eisenstein congruences on PGL(2)
- for $G =$U(prime), can guarantee cuspidal, non-endoscopic transfer by using G^* compact at a finite place