Distributions of local signs and murmurations

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Murmurations in Arithmetic Geometry and Related Topics
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Overview

I. Phenomena for global root numbers (Fricke eigenvalues) of modular forms

1. Root number bias
2. Correlation with initial Fourier coefficients
3. Murmurations

II. Phenomena for local root numbers (Atkin–Lehner eigenvalues) of modular forms (see *arXiv:2409.02338*)

1. Local root number bias
2. Correlation with initial Fourier coefficients
3. Murmurations

Takeaway: don't need root numbers for murmurations (for MFs)

Root number review

$f(z) = \sum a_n q^n \in S_k(N) = S_k(\Gamma_0(N))$ - newform

$L(s,f) = \sum \frac{a_n}{n^s} \times L_\infty(s,f) = \prod_{p \le \infty} L_p(s,f)$ - completed L-function

Global and local functional equations:

$$L(s,f) = \varepsilon(s,f) L(k-s,f)$$
$$L_p(s,f) = \frac{\varepsilon_p(s,f,\psi_p)}{\gamma_p(s,f,\psi_p)} L_p(k-s,f)$$

Global and local root numbers are ε-values at center $s = \frac{k}{2}$:

$$w_f = \prod_{p \le \infty} w_{f,p} = \pm 1, \quad w_{f,p} = 1 \text{ for } p \nmid N\infty$$

E.g.,

$$\varepsilon(s,f) = w_f N^{\frac{k}{2}-s}, \quad \text{so } w_f = \varepsilon(s, \tfrac{k}{2})$$

Root number review (cont'd)

Global and local root numbers are ε-values at center $s = \frac{k}{2}$:

$$w_f = \prod_{p \le \infty} w_{f,p} = \pm 1, \quad w_{f,p} = 1 \text{ for } p \nmid N\infty$$

Global root number controls (parity of) order of vanishing at $s = \frac{k}{2}$:

$$L(\tfrac{k}{2}, f) = w_f L(\tfrac{k}{2}, f)$$

so $w_f = -1 \implies L(\frac{k}{2}, f) = 0$

Local root numbers control signs of Fourier coefficients for $p \mid N$

$$a_p = \begin{cases} -w_{f,p} p^{\frac{k-1}{2}} & p \parallel N \\ 0 & p^2 \mid N, \end{cases}$$

(also control vanishing of "local periods")

Root number review (end)

Atkin–Lehner operators for $p \mid N$ and ∞:

- W_p - involution on $S_k(N)$ — locally $\begin{pmatrix} & 1 \\ p^{v_p(N)} & \end{pmatrix}$
- $W_\infty = (-1)^{\frac{k}{2}}$

Fricke involution on $S_k(N)$:

$$W = \prod_{p \mid N\infty} W_p$$

Global and local root numbers are eigenvalues of these involutions:

- $Wf = w_f f$
- $W_p f = w_{f,p} f$

Global root number bias: squarefree N

$S_k^{\mathrm{new}}(N)^\pm \subset S_k(N)$ - new subspace with global root number $w = \pm 1$

Equidistribution:
Iwaniec–Luo–Sarnak (2000): $\dim S_k^{\mathrm{new}}(N)^\pm = \frac{k-1}{24}\phi(N) + O((kN)^{\frac{5}{6}})$
$\implies w = +1$ (or -1) 50% of the time

Theorem (M. 2018, "strict bias")

1. $$\dim S_k^{\mathrm{new}}(N)^+ \geq \dim S_k^{\mathrm{new}}(N)^-,$$
   with equality only if (i) $\dim S_k^{\mathrm{new}}(N) = 0$*; (ii)* $N = 2, 3$*; (iii) or* $k = 2$ *and* $N = 37, 58$*.*
2. $$\dim S_k^{\mathrm{new}}(N)^+ - \dim S_k^{\mathrm{new}}(N)^- = c_N h_{\mathbb{Q}(\sqrt{-N})} - \delta_{k,2},$$
   where $c_N \in \left\{\frac{1}{2}, 1, 2\right\}$ *depends only on* $N \bmod 8$*.*

Global root number bias: non-squarefree N

- Pi–Qi (2021): $k \geq 4, N = M^3$, M squarefree
- Luo–Pi–Wu (2023): extends Pi–Qi to $k \geq 2$ and Hilbert modular forms
- M. (2023): N arbitrary
  — still get strict bias towards $w = +1$ *except* when $N \neq M^2$, M squarefree and $k \geq 12$
  — for fixed k, $\dim S_k^{\mathrm{new}}(N)^+ \geq \dim S_k^{\mathrm{new}}(N)^-$ for all but finitely many N.
  — for fixed M squarefree, get bias towards $w = (-1)^{k/2+\omega(M)}$ for $k \gg 0$

Proof (M. 2023) based on trace formula:

$$\mathrm{tr}_{S_k(N)} W = (-1)^{k/2} \frac{1}{2} \sum_{M|N, N/M \in \square} \mu(\sqrt{N/M}) H(-4M) + \delta_{k,2},$$

$H(D)$ - Hurwitz class number (weighted count of pos. def. BQFs/$\sim$)

Local root number bias

$N = p^r M$ $(r > 0,\ p \nmid M)$ — *for simplicity, assume M squarefree*

$S_k^{\mathrm{new}}(N)^{\pm_p} \subset S_k(N)$ - new subspace with local root number $w_p = \pm 1$

Equidistribution: $w_p = 1$ (or -1) 50% of the time

Theorem 1 (M.)

1. *If $r \neq 2$,*

$$(-1)^{\frac{k}{2}+\omega(M)}(\dim S_k^{\mathrm{new}}(N)^{+_p} - \dim S_k^{\mathrm{new}}(N)^{-_p}) \geq 0,$$

with equality iff $\left(\frac{-p^r}{q}\right) = 1$ for some prime $q \mid M$ when M odd.

2. *If $r = 2$, as $k + M \to \infty$,*

$$\dim S_k^{\mathrm{new}}(N)^{+_p} - \dim S_k^{\mathrm{new}}(N)^{-_p} \to -\infty.$$

— (1) w_p has a strict bias towards $(-1)^{\frac{k}{2}+\omega(M)}$

— also get formula for $\dim S_k^{\mathrm{new}}(N)^{+_p} - \dim S_k^{\mathrm{new}}(N)^{-_p}$

Bias of initial Fourier coefficients w.r.t. global root numbers

ℓ prime, $\ell \nmid N$

Theorem (M–Pharis (2022))

*N squarefree, $N \gg \ell$. Then**

$$\mathrm{tr}_{S_k^{\mathrm{new}}(N)^{\pm}} T_\ell \sim \pm \frac{1}{4} \ell^{\frac{k}{2}-1} H(4\ell N).$$

— use Yamauchi/Skoruppa–Zagier trace formula

— idea: trace formula for $T_\ell W_N$ is similar to that for $T_1 W_N$ when $\ell \ll N$

— a_ℓ tends to be *positive* for positive root number (N large)

— N squarefree can be relaxed

*Errata on webpage and `https://arxiv.org/abs/2409.02338`

Bias of initial Fourier coefficients w.r.t. local root numbers

ℓ prime, $\ell \nmid N$

$N = pM$ squarefree

Theorem (M.)

1. *If* $p \gg \ell$, *then* $\mathrm{tr}_{S_k^{\mathrm{new}}(pM)_p^{\pm}} T_\ell$ *has the same sign as* $\mathrm{tr}_{S_k^{\mathrm{new}}(pM)_p^{\pm}} T_\ell$ *if both are nonzero.*
2. *When* M *is odd, both traces are nonzero iff* $\left(\frac{-p}{q}\right) = \left(\frac{-p\ell}{q}\right) = 1$ *for each prime* $q \mid N$.

- $\implies$ signs of a_ℓ and w_p are positively (resp. negatively) correlated for $p \gg \ell$ when $w_p = +1$ (resp. $w_p = -1$) is the more common local root number
- recall $w_p = (-1)^{\frac{k}{2}+\omega(M)}$ is the more common local root number

Traces on newspaces

- $\mathrm{tr} T_1$ = dimension of $S_k^{\mathrm{new}}(N)$
- $\mathrm{tr} W = \mathrm{tr} W T_1 \rightsquigarrow$ root number bias
- $\mathrm{tr} W T_\ell$ for $\ell \ll N \rightsquigarrow$ initial correlation of a_ℓ with w
- $\mathrm{tr} W_p = \mathrm{tr} W_p T_1 \rightsquigarrow$ local root number bias
- $\mathrm{tr} W_p T_\ell$ for $\ell \ll N \rightsquigarrow$ initial correlation of a_ℓ with w_p

Question: Any correlation of a_ℓ with w or w_p for large ℓ?

Yes – murmurations

Murmurations

- elliptic curves — He–Lee–Oliver–Pozdnyakov (computations)
- modular forms — Sutherland (computations), Zubrilina (proofs for squarefree levels)
- other L-functions...

Murmurmation averages (global root numbers):

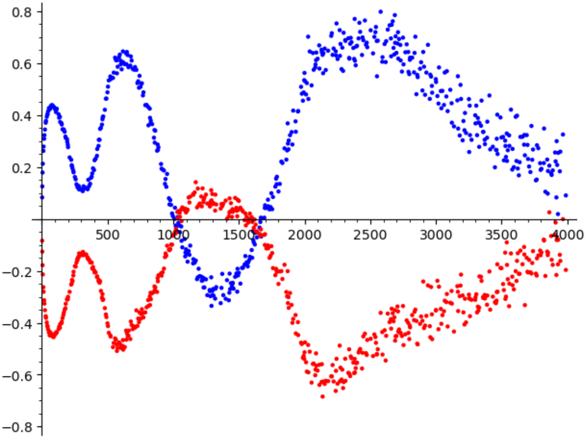
$$A_k^{\pm}(\ell; X, Y) = \ell^{1-\frac{k}{2}} \sideset{}{'}\sum_{X<N<Y} \sum_{f \in S_k(N)^{\pm} \text{ new}} a_\ell(f)$$

or

$$\begin{aligned} A_k(\ell; X, Y) &= A_k^+(\ell; X, Y) - A_k^-(\ell; X, Y) \\ &= \ell^{1-\frac{k}{2}} \sideset{}{'}\sum_{X<N<Y} \sum_{f \in S_k(N) \text{ new}} w_f a_\ell(f) = \ell^{1-\frac{k}{2}} \sideset{}{'}\sum_{X<N<Y} \mathrm{tr} W T_\ell \end{aligned}$$

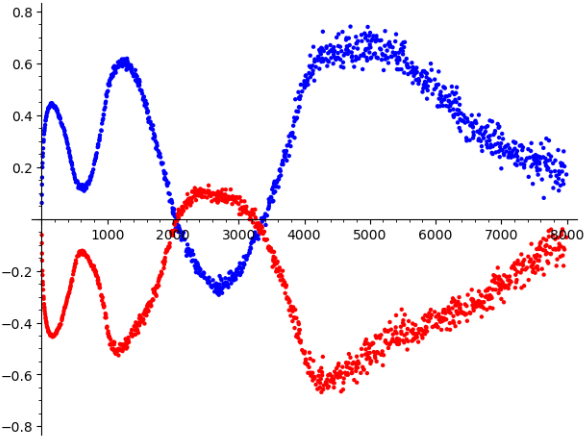
For simplicity: $\sum'$ means restrict to squarefree N coprime to ℓ

Weight 2 murmurations for global root numbers



$A_2^{\pm}(\ell, X, 2X)$ for $\ell \leq 4X$ with $X = 1000$

Weight 2 murmurations for global root numbers



$A_2^{\pm}(\ell, X, 2X)$ for $\ell \leq 4X$ with $X = 2000$

Murmurations for squarefree levels

Zubrilina:

- murmurmation averages tend to scale-invariant *murmuration functions* in ℓ/X:

$$A_k(\ell, X, \beta X) \to F_k(\ell/X, \beta)$$

  for $\beta > 1$ as $\ell, X \to \infty$ such that $\ell/X \to x$
- get murmuration density function ($\beta \to 0$) by considering $A_k(\ell, X, Y)$ with $Y = X + o(X)$
- properties of murmuration functions for $k = 2$

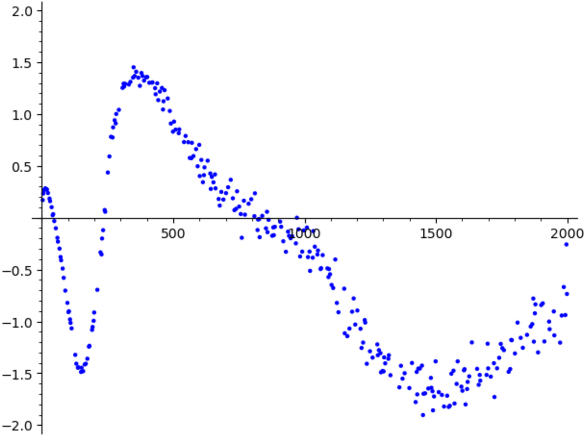
Murmurations for local root numbers?

First experiment (Type I):

- $\mathrm{tr} T_\ell W_Q \approx \mathrm{tr} T_\ell W_N$ if $Q \approx N$
- take $N = MQ$, M fixed, $Q \to \infty$ coprime to M
- then
$$A_k^Q(\ell; X, Y) = \ell^{1-\frac{k}{2}} \sum_{X<N=QM<Y}{}' \mathrm{tr} W_Q T_\ell$$
should behave similarly to
$$A_k(\ell; X, Y) = \ell^{1-\frac{k}{2}} \sum_{X<N<Y}{}' \mathrm{tr} W T_\ell$$

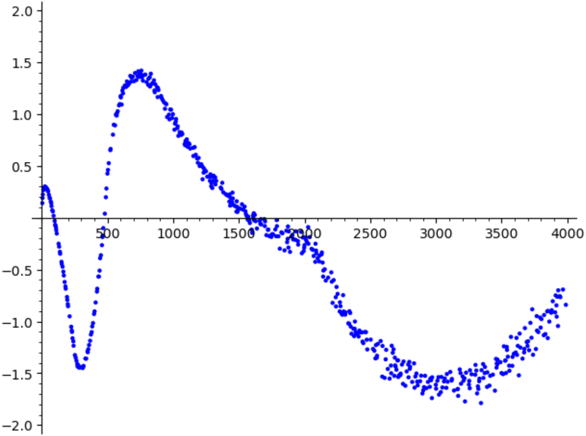
Note: $W_Q = \prod_{p|Q} W_p$
— can look at murmurations wrt single W_p by taking $Q = p$ (later)

Murmurations for local root numbers: weight 2, $N = 2Q$



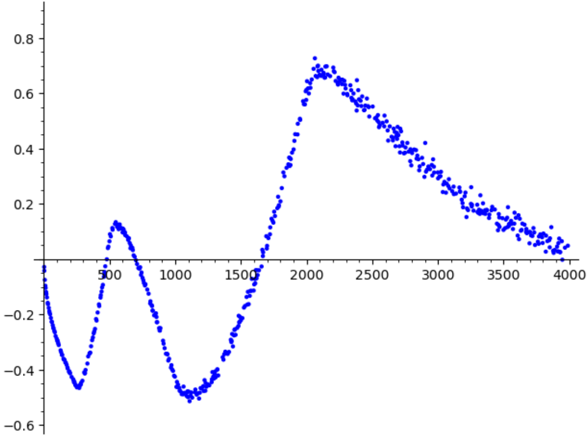
$\frac{1}{\sqrt{2}} A_2^Q(\ell, X, 2X)$ for $N = 2Q$ with $X = 1000$

Murmurations for local root numbers: weight 2, $N = 2Q$



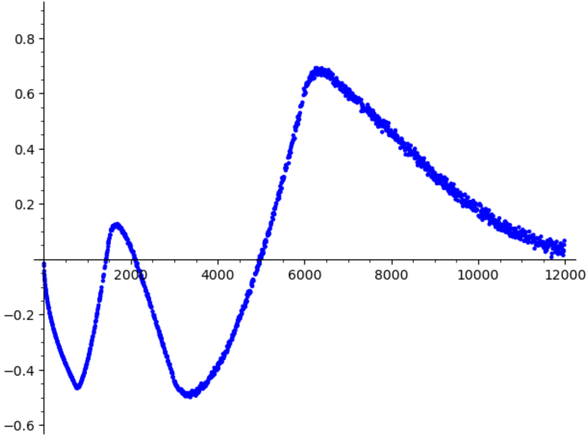
$\frac{1}{\sqrt{2}} A_2^Q(\ell, X, 2X)$ for $N = 2Q$ with $X = 2000$

Murmurations for local root numbers: weight 4, $N = 5Q$



$\frac{1}{\sqrt{5}} A_4^Q(\ell, X, 2X)$ for $N = 5Q$ with $X = 5000$

Murmurations for local root numbers: weight 4, $N = 5Q$



$\frac{1}{\sqrt{5}} A_4^Q(\ell, X, 2X)$ for $N = 5Q$ with $X = 15000$

Murmurations for fixed root numbers W_Q?

First experiment (Type I): $N = MQ$, M fixed, Q large successful

Theorem: limit as $\frac{\ell}{X} \to x$ tends to $c\sqrt{x} + \delta_{k=2} d$ when $x < \frac{1}{4M} - \varepsilon$

Proof: Average initial bias of a_ℓ's with w_Q

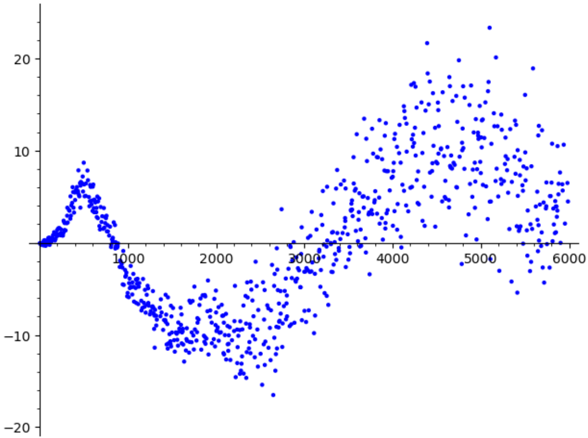
Second experiment (Type II):

- take $N = QM$, Q fixed, $M \to \infty$ coprime to Q
- similar to before, consider

$$A_k^Q(\ell; X, Y) = \ell^{1-\frac{k}{2}} \sideset{}{'}\sum_{X<N=QM<Y} \sqrt{M} \mathrm{tr} W_Q T_\ell$$

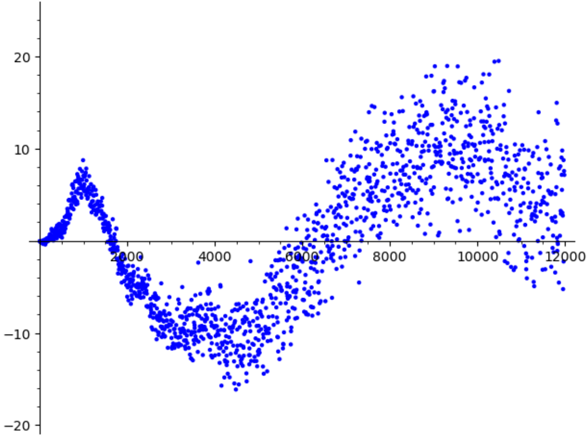
- note there is a weighting factor of $\sqrt{M} = \sqrt{N/Q}$ so averages $\not\to 0$

Murmurations for local root numbers: weight 2, $N = 2M$ $(Q = 2)$



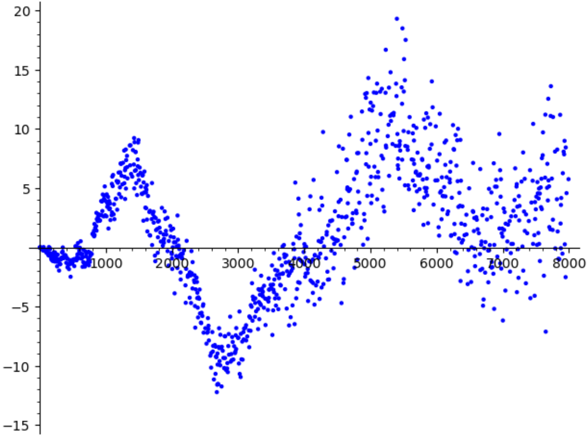
$A_2^Q(\ell, X, 2X)$ for $N = 2M$ with $X = 3000$ $(Q = 2)$

Murmurations for local root numbers: weight 2, $N = 2M$ $(Q = 2)$



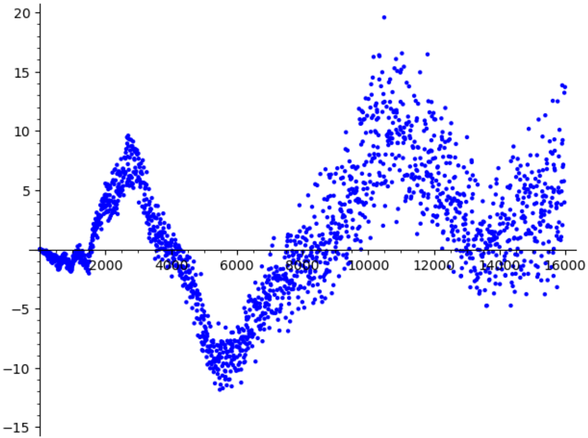
$A_2^Q(\ell, X, 2X)$ for $N = 2M$ with $X = 6000$ $(Q = 2)$

Murmurations for no root numbers: weight 2, $N = M$ $(Q = 1)$



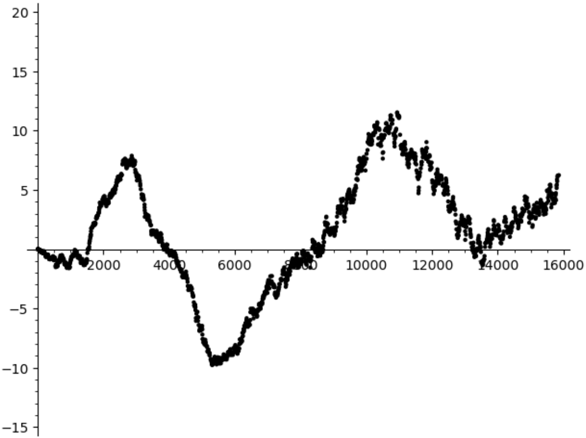
$A_2^Q(\ell, X, 2X)$ for $N = M$ with $X = 2000$ $(Q = 1)$

Murmurations for no root numbers: weight 2, $N = M$ $(Q = 1)$



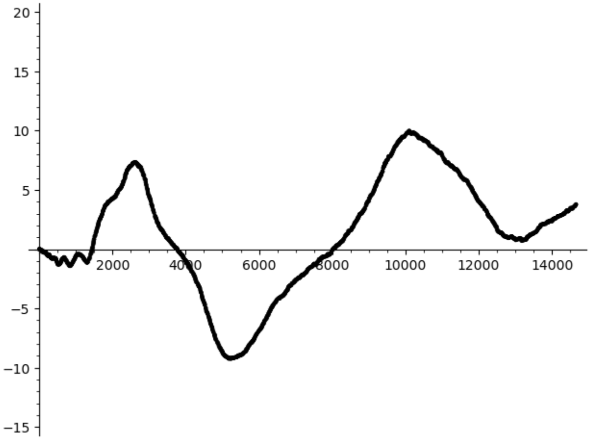
$A_2^Q(\ell, X, 2X)$ for $N = M$ with $X = 4000$ $(Q = 1)$

Smoothed murmurations for no root numbers: weight 2, $N = M$ ($Q = 1$)



$A_2^Q(\ell, X, 2X)$ for $N = M$ with $X = 4000$ ($Q = 1$) smoothed by averaging over ranges $[\ell, \ell + \ell^{\frac{1}{2}}]$

Smoothed murmurations for no root numbers: weight 2, $N = M$ ($Q = 1$)



$A_2^Q(\ell, X, 2X)$ for $N = M$ with $X = 4000$ ($Q = 1$) smoothed by averaging over ranges $[\ell, \ell + \ell^{\frac{3}{4}}]$

More general murmurations wrt W_Q?

First experiment (Type I): $N = MQ$, M fixed, Q large successful

(+ partial theorem)

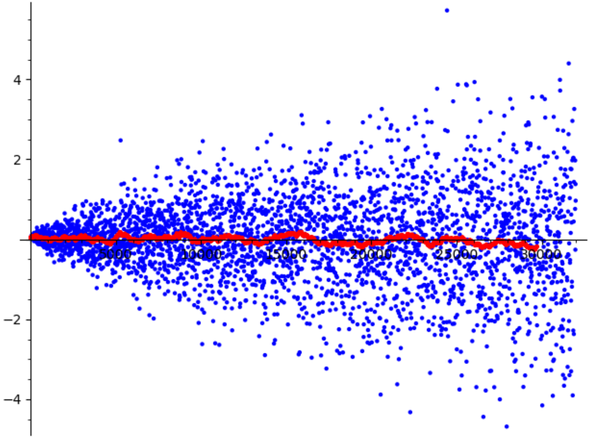
Second experiment (Type II): $N = QM$, Q fixed, M large successful (after weighting by $\sqrt{M}$ and possibly smoothing)

(no theorem: unbounded number of terms in trace formula)

Question: Can we work with more general families?

- can we choose $Q \mid N$ randomly, maybe with prescribe growth rate?
- does it matter if we use local root number bias?
(recall w_p is biased toward $(-1)^{\frac{k}{2}+\omega(M)}$ for $N = pM$ when M is squarefree odd).

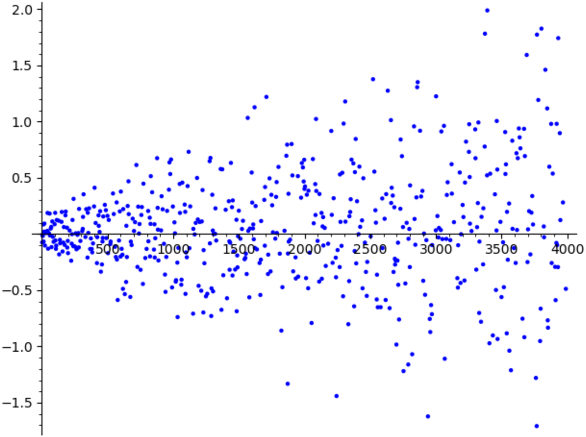
Murmuration averages for $N = QM$, $Q \approx \sqrt{N}$



restrict to N's with factor $Q \approx \sqrt{N}$

no apparent pattern

Signing murmation averages to bias $(-1)^{\frac{k}{2}+\omega(M)}$



$N = 5M$, weight 4, $p = Q = 5$ with weighted with sign $(-1)^{\frac{k}{2}+\omega(M)} w_p$
no apparent pattern

Expectations

— should take $N = QM$ along a sequence so that (N, Q) arithmetically well-behaved to see (interesting) murmations
— murmurations wrt w_p's not connected with initial bias to $(-1)^{\omega(M)}$ when $p \ll N$ (it really is only for $\ell \ll p$)

Conjecture 1

There are murmurations wrt $w_Q = \prod_{p|Q} w_Q$, which are scale invariant in ℓ/X, in the following situations. Here $N = QM$, $(Q, M) = 1$:

- *(Type I): M fixed, $Q \to \infty$ for MFs or ECs (possibly after smoothing for ECs)*
- *(Type II): Q fixed, $M \to \infty$ for MFs but not ECs (after weighting by $\sqrt{M}$ and possibly smoothing)*

— can take varying Q, M along all/squarefree/prime integers which are coprime to the other

Comparing murmurations wrt different W_Q's

For simplicity: $N = pq$ (p, q distinct primes)

— could look at murmurations averages for all AL-operators: W_1, W_p, W_q, W_{pq}

Better:

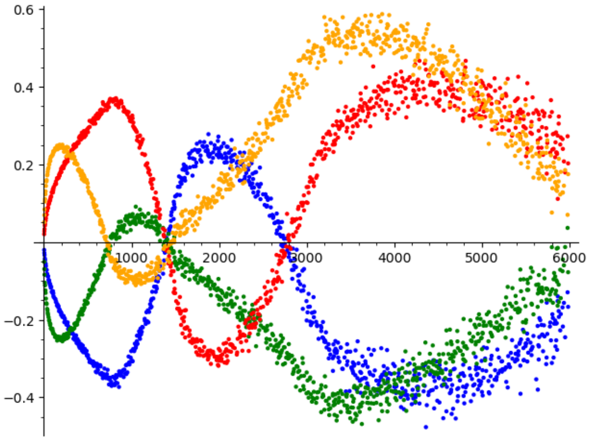
— look at murmurations with fixed AL-signs: $\varepsilon = (w_p, w_q)$: ++, −−, +−, −+

— analogous to plotting murmurations for fixed root numbers (red + blue plots), rather than difference

related by linear combinations: $\mathrm{tr}_{S_k^{\mathrm{new}}(pq)^{\varepsilon}} T_\ell =$

$$\frac{1}{4}\left(\mathrm{tr} T_\ell W_1 + w_p \mathrm{tr} T_\ell W_p + w_q \mathrm{tr} T_\ell W_q + w_p w_q \mathrm{tr} T_\ell W_{pq}\right)$$

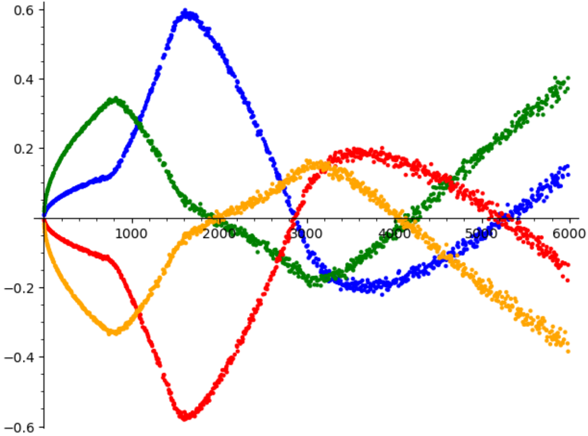
Murmurations on AL-eigenspaces for $N = 2q$: weight 2



$w_q = w_{pq}$

++	−−	$w_{pq} = +1$
+−	−+	$w_{pq} = -1$

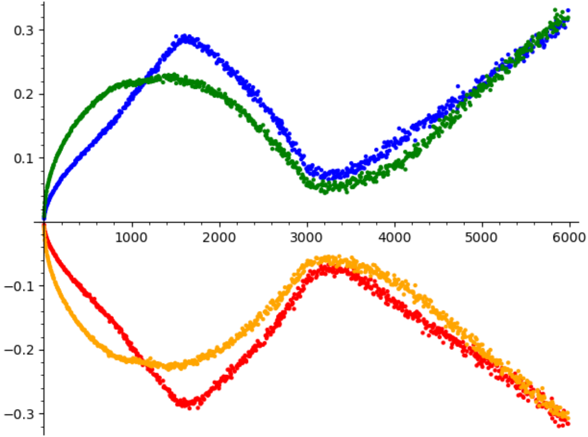
Murmurations on AL-eigenspaces for $N = 2q$: weight 4



$w_q = w_{pq}$

++	−−	$w_{pq} = +1$
+−	−+	$w_{pq} = -1$

Murmurations on AL-eigenspaces for $N = pq$: weight 4



$w_q = w_{pq}$

++	−−	$w_{pq} = +1$
+−	−+	$w_{pq} = -1$

Murmurations on AL-eigenspaces

Conjecture 2

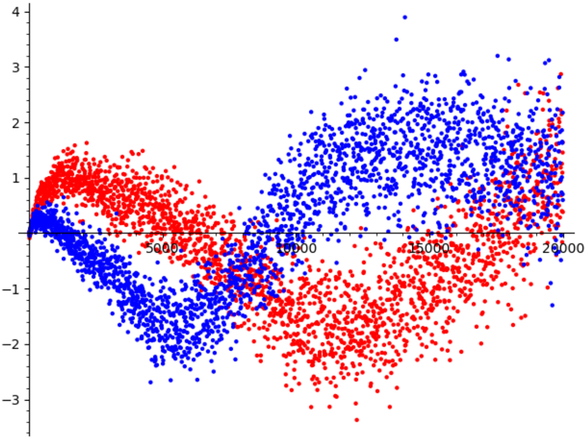
Taking $N = p_1 p_2 \dots p_r \to \infty$ *with* $p_1 < p_2 \cdots < p_r$*, one sees murmurations on each Atkin–Lehner eigenspace* $S_k^{\mathrm{new}}(pq)^{\varepsilon}$ *(*$\varepsilon = (w_{p_1}, \dots, w_{p_r})$*) in either of the following cases:*

- N *runs over all squarefree products of* r *primes (all* p_i*'s vary)*
- N *runs over all squarefree multiples of* $p_1 \dots p_m$*,* $m < r$ *(only* $p_{m+1}, \dots, p_r$*) vary*

When $N = pq$ $(p < q)$: expect AL-eigenspace murmurations are dominated by those for global root number and those for W_q

Reason: look at $\underbrace{\mathrm{tr} T_\ell W_1}_{\to 0} + \underbrace{\mathrm{tr} T_\ell W_p}_{\to 0} + w_q \mathrm{tr} T_\ell W_q + w_p w_q \mathrm{tr} T_\ell W_{pq}$

W_Q murmurations for elliptic curves for $N = 2Q$



W_Q murmurations

$W = -W_{2Q}$ murmurations