Rational genus 2 curves with real multiplication

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(Hyper)elliptic curves of low genus

Consider (hyper)elliptic curves over $\mathbb{Q}$:

<u>**Genus** 1</u>: Elliptic curves $E : y^2 = x^3 + ax + b$

- $\mathbb{C}$-isomorphism determined by j-invariant $1728\frac{4a^3}{4a^3+27b^2}$
- (twist classes/$\mathbb{Q}$) parametrized by modular curve $\mathcal{M}_1(\mathbb{Q})$
- correspond to rational newforms: $f = \sum a_n q^n \in S_2(N) \quad (a_n \in \mathbb{Z})$

<u>**Genus** 2</u>: Hyperelliptic $C : y^2 = c_6x^6 + c_5x^5 + \cdots + c_1x + c_0$

- $\mathbb{C}$-isomorphism determined Igusa–Clebsch invariants $(I_2 : I_4 : I_6 : I_{10}) \in \mathbb{P}^3_{2,4,6,10}$
- over $\mathbb{C}$, parametrized by modular 3-fold

$$\begin{aligned} \mathcal{M}_2 \subset \quad & \mathcal{A}_2 \simeq \mathbb{P}^3_{2,4,6,10} \\ C \to \quad & A = \mathrm{Jac}(C) \end{aligned}$$

Genus 2 curves: endomorphism types

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- modularity classification

$\mathrm{End}(A)$	modularity	type
$\mathbb{Z}$	deg 2 Siegel modular form*	typical
quad. order $\mathcal{O}$	$f = \sum a_n q^n \in S_2(N) \quad (a_n \in \mathcal{O})$	GL(2)
matrix/split $\mathcal{O}$	$E_1 \times E_2$	GL(2)

*conjectural

Multiplication

C - genus 2 curve/$\mathbb{Q}$, $A = \mathrm{Jac}(C)$

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$\mathbb{Z}$	deg 2 Siegel modular form*	typical
quad. order $\mathcal{O}$	$f = \sum a_n q^n \in S_2(N) \quad (a_n \in \mathcal{O})$	GL(2)
matrix/split R	$E_1 \times E_2$	GL(2)

Definition 1

Let $\mathcal{O}$ be an order in real/complex quadratic field. We say C (or A) has **real/complex multiplication by** $\mathcal{O}$ if $\mathcal{O} \hookrightarrow \mathrm{End}^\dagger(A)$.
($\dagger$ - Rosati involution)

— $D > 0$: fundamental discriminant
— Say C (or A) **has RM** D if it has real multiplication by $\mathcal{O}_D := \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$

Curves with RM D

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— C (or A) **has RM** D if it has real multiplication by $\mathcal{O}_D := \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$

Question

1. *How to construct models of genus 2 curves with RM D?*
2. *How to parametrize genus 2 curves with RM D?*

Known constructions:

- Mestre: RM 5 and RM 8 families (2 parameter)
- Brumer: RM 5 family (3 parameter)
- Bending: RM 8 family (3 parameter)

Drawbacks:

— Mestre's and Brumer's families not (?) universal/generic over $\mathbb{Q}$
— not clear when curves in family are isomorphic (even over $\mathbb{C}$)
(parameter space 2-dim)
— constructions are very specific to RM 5 and 8

Generic models for RM D

Theorem 1 (Cowan–M, to appear)

Give generic models for genus 2 curves/$\mathbb{Q}$ with RM 5.

Theorem 2 (Cowan–Frengley–M, in prep.)

Give generic models for genus 2 curves/$\mathbb{Q}$ with RM D for $D = 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 44, 53, 61$.

Example 1 (Generic model for $D = 12$)

Generic RM 12 model: $y^2 = N_{L/\mathbb{Q}} h(x)$ *in parameters* a, b, c, *where* $g(r) = r^3 - 3(a^2 - 3b^2)r + 2(a^2 - 3b^2)$, $L = \mathbb{Q}[r]/g(r)$, *and*

$$h(x) = (r^2 + (2a - 3b)r - (a^2 + 2a - 3b^2 - 3bc - 3b))x^2 - 6((a-2b)r - ac - a + 2b)x - 3(r^2 - (2a-3b)r - (a^2 - 2a - 3b^2 + 3bc + 3b)).$$

Generic models for RM D II

Theorem 3 (Cowan–Frengley–M, in prep.)

Give generic models for genus 2 curves/$\mathbb{Q}$ with RM D for $D = 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 44, 53, 61.$

Example 2 (Generic model for $D = 17$)

Generic RM 17 model: $y^2 = g(x)h(x)$ *in parameters* a, b, c*, where*

$$g(x) = (a^2 - 8ab + 4a - 9b^2 - 6b + 3)x^3 + 3(7ab - 3a + 7b^2 + 4b - 3)x^2 + 4(a^2 - 7ab + 3a - 4b^2)x + 4(3ab - a + b^2 - b),$$

$$h(x) = 4(a^2b + 5ab^2 - 7ab + 2a - 6b^2 + 2b)x^3 + 4(6a^2b - 2a^2 - 12ab^2 + 11ab - 3a + 14b^2 - 6b)x^2 + (4a^3 - 34a^2b + 16a^2 + 38ab^2 - 43ab + 9a - 43b^2 + 36b - 9)x + 12a^2b - 4a^2 - 10ab^2 + 14ab - 4a + 11b^2 - 14b + 3.$$

Hilbert modular surfaces

$D > 0$ - fundamental discriminant, $\mathcal{O}_D = \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$

$Y_-(D) = \overline{(\mathfrak{H}^+ \times \mathfrak{H}^-)}/\mathrm{SL}_2(\mathcal{O}_D)$ – Hilbert modular surface

$Y_-(D)$ - **coarse** moduli space for (A, ι), where

— A: principally polarized abelian surface A with RM D

— $\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}(A)^\dagger$

Coarse $\implies$

$$\{\text{genus 2 } C \text{ with RM } D\} \to Y_-(D)(\mathbb{Q}) \setminus \{P : I_{10} = 0\}$$

but not $\leftarrow$ (**Mestre obstruction**)

Elkies–Kumar (2014): Birational models for $Y_-(D)/\mathbb{Q}$ for $D < 100$

Parametrizing RM D curves

Rational Surfaces:

$Y_-(D)$ is birational to $\mathbb{A}^2_{m,n} \iff D = 5, 8, 12, 13, 17$

Theorem 4 (Cowan–Frengley–M)

Generically, $(m,n) \mapsto Y_-(D)(\mathbb{Q})$ *corresponds to a rational genus 2 curve with RM* $D \iff p_D(m,n) \in \mathbb{Q}$ *is a norm from* $\mathbb{Q}(\sqrt{D})$*, where*

$$p_D(m,n) = \begin{cases} m^2 - 5n^2 - 5 & D = 5 \\ m + 1 & D = 8 \\ -27m^2 + n^2 + 27 & D = 12 \\ 1803m^2 - 72mn + n^2 + 3168m - 1440n - 768 & D = 13 \\ 1 & D = 17. \end{cases}$$

Non-rational Surfaces:

Analogous result for $D = 21, 24, 28, 29, \mathbf{33}, 37, 44, 53, 61$ **or** $D \equiv 1 \bmod 8$

Outline of proofs

1. Begin with Elkies–Kumar model for $Y_-(D)$
2. Mestre obstruction: Mestre conic $L(P) = L(g,h)$ has a point, $P \in Y_-(D)$ (g, h: coordinates on quotient of $Y_-(D)$)
3. Use nicer version of Mestre conic $L(g,h)$ for Igusa–Clebsch/Elkies–Kumar invariants
4. "Algorithm" to minimize and reduce $L(g,h)$ (+ general argument that Mestre obstruction vanishes when $D \equiv 1 \bmod 8$) $\implies$ Theorem 4
5. Use transformations from algorithm + Mestre's method to give models for curves $\implies$ Theorems 1, 2
6. When possible, apply various tricks to simplify models $\implies$ Examples 1, 2

Example: $D = 8$

Mestre conic $L(g,h)$ for $D = 8$:

$$L = (256g^3 - 32g^2h - 207gh^2 + 81h^3 - 2312g^2 + 676gh - 162h^2 + 88g + 13h + 6)\,x_0^2$$
$$+(-512g^4 + 64g^3h + 414g^2h^2 - 162gh^3 - 96g^3 + 2028g^2h + 549gh^2 + 336g^2 - 162gh - 18h^2 +$$
$$+(256g^5 - 32g^4h - 207g^3h^2 + 81g^2h^3 + 608g^4 - 904g^3h - 2412g^2h^2 + 576g^3 - 876g^2h + 18gh^2$$
$$+(14336g^5 + 16640g^4h - 13896g^3h^2 - 10368g^2h^3 + 5832gh^4 - 61568g^4 - 137312g^3h + 48960g^2$$
$$+(-14336g^6 - 16640g^5h + 13896g^4h^2 + 10368g^3h^3 - 5832g^2h^4 - 27392g^5 + 51232g^4h - 42480$$
$$+(200704g^7 + 491008g^6h + 104976g^5h^2 - 395280g^4h^3 - 104976g^3h^4 + 104976g^2h^5 - 170496g^6$$

Coefficient degree: 7

Discriminant:

$$4100625h^2g^2(16g^3 + 32g^2h + 16gh^2 + 24g^2 - 40gh + 12g - h + 2) \cdot (256g^3 - 288g^2h + 81gh^2 + 256g^2 - 48gh + 64g - 16h)^2$$

Example: Minimization for $D = 8$

Discriminant of $L(g,h)$:

$$4100625h^2g^2(16g^3+32g^2h+16gh^2+24g^2-40gh+12g-h+2)$$
$$\cdot(256g^3-288g^2h+81gh^2+256g^2-48gh+64g-16h)^2$$

Step	Minimize at	New factors	Coeff deg
0			7
1	∞ (removes factors of $3, g, h$)		4
2	$256g^3-288g^2h+81gh^2$ $+256g^2-48gh+64g-16h$	$(6g+1)^2$	3
3	∞		2
4	$6g+1$ (removes powers of 2)		2

$$\ldots \rightsquigarrow \qquad L(g,h) \equiv x_0^2 - 2x_1^2 - (4g+4h-7)x_2^2$$