Central L-values and functorial transfer:

nonvanishing and congruences

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Motivation

$\mathcal{E}/\mathbb{Q}$ - elliptic curve of conductor N
e.g., $\mathcal{E}_{17a1} : y^2 + xy + y = x^3 - x^2 - x - 14$ with $N = 17$

Functional equation:

$$\begin{aligned} L(1-s,\mathcal{E}) &= \varepsilon(s,\mathcal{E})L(s,\mathcal{E}) \\ L(\tfrac{1}{2},\mathcal{E}) &= \varepsilon(\tfrac{1}{2},\mathcal{E})L(\tfrac{1}{2},\mathcal{E}) \\ \varepsilon(\tfrac{1}{2},\mathcal{E}) &= \pm 1 \text{ (root number)} \end{aligned}$$

Conjecture (weak BSD)
$|\mathcal{E}(\mathbb{Q})|$ *is finite* $\iff L(\frac{1}{2},\mathcal{E}) \neq 0.$
Kolyvagin: true if $\mathrm{ord}_{s=\frac{1}{2}} L(s,\mathcal{E}) \leq 1$

General Problem

Problem: For (automorphic, self-dual) L-functions, study when $L(\frac{1}{2}, \pi) \neq 0$

Philosophy: Expect $L(\frac{1}{2}, \pi) = 0$ 50% of the time and $L(\frac{1}{2}, \pi) \neq 0$ 50% of the time (100% of time when $\varepsilon = +1$)

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A concrete conjecture
d - fundamental discriminant
$\chi_d(n) = \left(\frac{d}{n}\right)$ - quadratic Dirichlet character
$\mathcal{E}^d$ - quadratic twist of $\mathcal{E}$ by χ_d

Conjecture (Goldfeld)

Given $\mathcal{E}$, $L(\frac{1}{2}, \mathcal{E}^d) \neq 0$ for 50% of d and $L(\frac{1}{2}, \mathcal{E}^d) = 0$ (in fact $\mathrm{ord}_{s=\frac{1}{2}} L(s, \mathcal{E}^d) = 1$) for 50% of d (ordered by absolute value)

Some partial results

Conjecture (Goldfeld)
Give $\mathcal{E}$, $L(\frac{1}{2}, \mathcal{E}^d) \neq 0$ for 50% of d and $L(\frac{1}{2}, \mathcal{E}^d) = 0$ (in fact $\operatorname{ord}_{s=\frac{1}{2}} L(s, \mathcal{E}^d) = 1$) for 50% of d (ordered by absolute value)

Note: $\varepsilon(\frac{1}{2}, \mathcal{E}^d) = +1$ for 50% of d

James, Vatsal, ... : There exists $\mathcal{E}$ such that $L(\frac{1}{2}, \mathcal{E}^d) \neq 0$ (resp. $\operatorname{ord}_{s=\frac{1}{2}} L(s, \mathcal{E}^d) = 1$) for a positive proportion of d.

Ono–Skinner, ... : $L(\frac{1}{2}, \mathcal{E}^d) \neq 0$ for a "large zero proportion" of d

me? : $\mathcal{E} = \mathcal{E}_{17a1}$ and $d < 0$ prime. Then

$$L(\tfrac{1}{2}, \mathcal{E}^d) \neq 0 \iff \left(\frac{d}{17}\right) = -1 \quad \text{(50\% of time)}$$

Idea of proof

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Say $\mathcal{E} \leftrightarrow f \leftrightarrow \pi$
f - modular form
π - automorphic representation

Step 1. Relate $L(\frac{1}{2}, \mathcal{E}^d) = L(\frac{1}{2}, f \otimes \chi_d)$ to a **period** $P_d(\phi)$ on some $\pi' \overset{\mathrm{JL}}{\leftrightarrow} \pi$ (Waldspurger, uses functoriality).

Step 2. Show $f \equiv E_{2,17} \mod 2$ (M., also uses functoriality), where $E_{2,17}$ is the weight 2, level 17 Eisenstein series

Step 3. Conclude $L^{\mathrm{alg}}(\frac{1}{2}, f \otimes \chi_d) \equiv *L^{\mathrm{alg}}(\frac{1}{2}, E_{2,17} \otimes \chi_d) \mod 2$ (Mazur, Vatsal, M., ...)

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Need to generalize:

1. relate L-values to periods
2. Hecke eigenvalue congruence with Eisenstein series
3. deduce nonvanishing of L-values (easy, won't elaborate)

Central L-values and periods: $\mathrm{GL}(2)$

π cusp aut rep of $G = \mathrm{PGL}(2) \simeq \mathrm{SO}(2,1)$,
π' cusp aut rep of $G' = PB^\times \simeq \mathrm{SO}(3)$, B - quaternion algebra,
$\pi' \leftrightarrow \pi$

$H = K^1/\{\pm 1\} \simeq \mathrm{SO}(2) \subset G'$, $K/\mathbb{Q}$ - quadratic ($\leftrightarrow d$)
period $P_H : \pi' \to \mathbb{C}$, $\quad P_H(\phi;\pi') = \int_H \phi(h)\, dh$
(finite sum if G' compact)

Theorem (Waldspurger)

$$L(\tfrac{1}{2}, \pi \times 1_H) \neq 0 \iff P_H(\phi;\pi') \not\equiv 0 \text{ for some } \pi' \leftrightarrow \pi, \phi \in \pi'$$

and in fact

$$|P_H(\phi;\pi')|^2 = c(\phi,\pi',H) L(\tfrac{1}{2}, \pi \times 1_H).$$

Here

$$L(\tfrac{1}{2}, \pi \times 1_H) = L(\tfrac{1}{2}, \pi) L(\tfrac{1}{2}, \pi \otimes \chi_d).$$

Central L-values and periods: $\mathrm{SO}(2n+1)$

π cusp aut rep of $G = \mathrm{SO}(n+1,n)$
π' cusp aut rep of $G' = \mathrm{SO}(2n+1)$, $\quad \pi' \leftrightarrow \pi$

$H = \mathrm{SO}(2m)$ (compact) $\subset G'$
period $P_H : \pi' \to \mathbb{C}$, $\quad P_H(\phi;\pi') = \int_{HN} \phi(hn)\psi_N(n)\, dh\, dn$

Conjecture (Gan–Gross–Prasad; Ichino–Ikeda–Liu)

$$L(\tfrac{1}{2}, \pi \times 1_H) \neq 0 \iff P_H(\phi;\pi') \not\equiv 0 \text{ for some } \pi' \leftrightarrow \pi, \phi \in \pi'.$$

Moreover

$$|P_H(\phi;\pi')|^2 = c(\phi,\pi',H)L(\tfrac{1}{2}, \pi \times 1_H).$$

Theorem (Furusawa–Morimoto)

When $m = 1$, one direction and formula is true.

Theorem (Jiang–Zhang)

One direction of nonvanishing is true.

Eisenstein congruences for $\mathrm{GL}(2)$

Theorem (Mazur, 1977)

N - prime, $p | \frac{N-1}{12}$, $p \neq 2$, $\mathfrak{p} | p$ in $\overline{\mathbb{Q}}$
$E_{2,N}$ - Eisenstein series of weight 2 and level N
There exists a newform $f \in S_2(N)$ such that

$$f \equiv E_{2,N} \mod \mathfrak{p}$$

- Ribet ($\sim$2010, unpub) extended to squarefree N assuming $p \geq 5$, $p \nmid N$
- me: extended to N not a square, any p; also for HMFs
- Billerey–Menares (2016): analogue for higher weight in prime level (p large)

Example of a congruence

Claim: $f \in S_2(17)$ satisfies $f \equiv E_{2,17} \mod 2$.

Proof.
$M_2(17) \simeq$ DefQuatMF(17) (JL corr)
DefQuatMF(17) $= \{\phi : \mathcal{O}(17) \to \mathbb{C}\}$, $\mathcal{O}(17) = \{\mathcal{I}_1, \mathcal{I}_2\}$

ϕ	$\phi(\mathcal{I}_1)$	$\phi(\mathcal{I}_2)$
$E_{2,17} \leftrightarrow \phi_0$	1	1
$f \leftrightarrow \phi_1$	3	-1

Note $\phi_0 \equiv \phi_1 \mod 2$. □

More general congruences

G - reductive group/$\mathbb{Q}$ that $G(\mathbb{R})$ compact

(me) G form of $\mathrm{PGL}(2)$, i.e., $G = PB^\times$:

- if $p|m(\mathcal{O})$, get $\phi \equiv 1 \mod p$ for quaternionic form ϕ
- by JL corr, get Eisenstein congruence

(me–Wakatsuki) G reductive:

- if $p|m(\Lambda)$, get $\phi \equiv 1 \mod p$ for algebraic modular form ϕ
- will give Eisenstein congruences for split form G' once we know generalized JL corr (e.g., for genus 2 Siegel modular forms when $G \sim \mathrm{PGSp}(4)$)