Biases of modular forms from the trace formula

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Weight 2 forms on the LMFDB

Label	Dim	A	Field	Traces				Fricke sign	q-expansion
				a_2	a_3	a_5	a_7		
11.2.a.a	1	0.088	$\mathbb{Q}$	-2	-1	1	-2	$-$	$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots$
14.2.a.a	1	0.112	$\mathbb{Q}$	-1	-2	0	1	$-$	$q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + \cdots$
15.2.a.a	1	0.120	$\mathbb{Q}$	-1	-1	1	0	$-$	$q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + \cdots$
17.2.a.a	1	0.136	$\mathbb{Q}$	-1	0	-2	4	$-$	$q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + \cdots$
19.2.a.a	1	0.152	$\mathbb{Q}$	0	-2	3	-1	$-$	$q - 2q^3 - 2q^4 + 3q^5 - q^7 + q^9 + 3q^{11} + \cdots$
20.2.a.a	1	0.160	$\mathbb{Q}$	0	-2	-1	2	$-$	$q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + \cdots$
21.2.a.a	1	0.168	$\mathbb{Q}$	-1	1	-2	-1	$-$	$q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + \cdots$
23.2.a.a	2	0.184	$\mathbb{Q}(\sqrt{5})$	-1	0	-2	2	$-$	$q - \beta q^2 + (-1 + 2\beta)q^3 + (-1 + \beta)q^4 + \cdots$
24.2.a.a	1	0.192	$\mathbb{Q}$	0	-1	-2	0	$-$	$q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + \cdots$
26.2.a.a	1	0.208	$\mathbb{Q}$	-1	1	-3	-1	$-$	$q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 + \cdots$
26.2.a.b	1	0.208	$\mathbb{Q}$	1	-3	-1	1	$-$	$q + q^2 - 3q^3 + q^4 - q^5 - 3q^6 + q^7 + \cdots$
27.2.a.a	1	0.216	$\mathbb{Q}$	0	0	0	-1	$-$	$q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots$
29.2.a.a	2	0.232	$\mathbb{Q}(\sqrt{2})$	-2	2	-2	0	$-$	$q + (-1 + \beta)q^2 + (1 - \beta)q^3 + (1 - 2\beta)q^4 +$

— a_2, a_3's start off negative

— root number tends to be $+1$

Weight 4 forms on the LMFDB

Label	Dim	A	Field	Traces a_2	a_3	a_5	a_7	Fricke sign	q-expansion
5.4.a.a	1	0.295	$\mathbb{Q}$	-4	2	-5	6	$+$	$q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + \cdots$
6.4.a.a	1	0.354	$\mathbb{Q}$	-2	-3	6	-16	$+$	$q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 + \cdots$
7.4.a.a	1	0.413	$\mathbb{Q}$	-1	-2	16	-7	$+$	$q - q^2 - 2q^3 - 7q^4 + 2^4q^5 + 2q^6 + \cdots$
8.4.a.a	1	0.472	$\mathbb{Q}$	0	-4	-2	24	$+$	$q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + \cdot$
9.4.a.a	1	0.531	$\mathbb{Q}$	0	0	0	20	$+$	$q - 8q^4 + 20q^7 - 70q^{13} + 2^6q^{16} + \cdots$
10.4.a.a	1	0.590	$\mathbb{Q}$	2	-8	5	-4	$+$	$q + 2q^2 - 8q^3 + 4q^4 + 5q^5 - 2^4q^6 + \cdots$
11.4.a.a	2	0.649	$\mathbb{Q}(\sqrt{3})$	2	-2	2	20	$+$	$q + (1+\beta)q^2 + (-1-4\beta)q^3 + (-4+2$
12.4.a.a	1	0.708	$\mathbb{Q}$	0	3	-18	8	$+$	$q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 6^2q^{11} + \cdot\cdot$
13.4.a.a	1	0.767	$\mathbb{Q}$	-5	-7	-7	-13	$-$	$q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 + \cdots$
13.4.a.b	2	0.767	$\mathbb{Q}(\sqrt{17})$	1	5	-3	-9	$+$	$q + \beta q^2 + (4-3\beta)q^3 + (-4+\beta)q^4 + \cdot\cdot$
14.4.a.a	1	0.826	$\mathbb{Q}$	-2	8	-14	-7	$+$	$q - 2q^2 + 8q^3 + 4q^4 - 14q^5 - 2^4q^6 + \cdots$
14.4.a.b	1	0.826	$\mathbb{Q}$	2	-2	-12	7	$+$	$q + 2q^2 - 2q^3 + 4q^4 - 12q^5 - 4q^6 + \cdots$
15.4.a.a	1	0.885	$\mathbb{Q}$	1	3	5	-24	$+$	$q + q^2 + 3q^3 - 7q^4 + 5q^5 + 3q^6 + \cdots$
15.4.a.b	1	0.885	$\mathbb{Q}$	3	-3	-5	20	$+$	$q + 3q^2 - 3q^3 + q^4 - 5q^5 - 9q^6 + \cdots$

— a_2's start off negative

— root number tends to be $+1$

Seeking explanations...

Farmer–Koutsoliotas (2016): *The second Dirichlet coefficient starts out negative*
— L-functions from nothing approach, transient phenomenon
— similar argument says root numbers start out $+1$, known to be transient for elliptic curves

Theorem 1 (strict bias of root numbers, M. 2018)

Fix $k \in 2\mathbb{N}$, N *squarefree. Then*

$$\dim S_k^{\mathrm{new}}(N)^+ \geq \dim S_k^{\mathrm{new}}(N)^-,$$

with equality only if (i) $\dim S_k^{\mathrm{new}}(N) = 0$*; (ii)* $N = 2, 3$*; (iii) or* $k = 2$ *and* $N = 37, 58$*. Moreover,*

$$\dim S_k^{\mathrm{new}}(N)^+ - \dim S_k^{\mathrm{new}}(N)^- = c_N h_{\mathbb{Q}(\sqrt{-N})} - \delta_{k,2},$$

where $c_N \in \left\{\frac{1}{2}, 1, 2\right\}$ *depends only on* $N \bmod 8$*.*

Root number bias: timeline

- M. (2018): N squarefree $\implies$

$$\dim S_k^{\text{new}}(N)^+ - \dim S_k^{\text{new}}(N)^- = c_N h_{\mathbb{Q}(\sqrt{-N})} - \delta_{k,2} \geq 0$$

  — uses trace formula for Atkin–Lehner operators

- Pi–Qi (2021): $k \geq 4, N = M^3$, M squarefree $\implies$

$$\dim S_k^{\text{new}}(N)^+ - \dim S_k^{\text{new}}(N)^- = c_N \phi(N) h_{\mathbb{Q}(\sqrt{-N})} - \delta_{k,2} \geq 0$$

  — uses Petersson trace formula weighted by root numbers

- Luo–Pi–Wu (2023): extends Pi–Qi to $k \geq 2$ and Hilbert modular forms (explicit over $F = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})$)
  — uses Jacquet–Zagier trace formula

- M.: N arbitrary
  — uses trace formula for Atkin–Lehner operators

Root number bias: general levels

Theorem 2 (M.)

1. *If (a) $N \neq M^2$, M squarefree, or (b) $k \leq 10$ or $k = 14$, then there is a strict root number bias towards $+1$:*

$$\dim S_k^{\mathrm{new}}(N)^+ \geq \dim S_k^{\mathrm{new}}(N)^-.$$

2. *If $N \neq M^2, 2M^2, 3M^2, 4M^2$, M squarefree, then*

$$\dim S_k^{\mathrm{new}}(N)^+ - \dim S_k^{\mathrm{new}}(N)^- = b_N h_{\mathbb{Q}(\sqrt{-N})} - \delta_{k,2}\delta_{N,1} \geq 0.$$

3. *For any fixed k, $\dim S_k^{\mathrm{new}}(N)^+ \geq \dim S_k^{\mathrm{new}}(N)^-$ for all but finitely many N.*
4. *If $N = M^2$, M squarefree and $k \gg N$, then*

$$(-1)^{k/2+\omega(M)}(\dim S_k^{\mathrm{new}}(N)^+ - \dim S_k^{\mathrm{new}}(N)^-) \geq 0.$$

Root number bias: proof ingredients

W_N - Fricke involution on $S_k(N)$
If f is a newform with root number $w_f = \pm 1$,

$$W_N f = (-1)^{k/2} w_f f$$

Hence

$$(-1)^{k/2} \mathrm{tr}_{S_k^{\mathrm{new}}(N)} W_N = \dim S_k^{\mathrm{new}}(N)^+ - \dim S_k^{\mathrm{new}}(N)^-$$

Yamauchi (1973)/Skoruppa–Zagier (1988): trace formula for
$\mathrm{tr}_{S_k(N)} W_M T_\ell$
$\ell = 1$, $M = N > 4$ relatively simple:

$$\mathrm{tr}_{S_k(N)} W_N = (-1)^{k/2} \frac{1}{2} \sum_{M | N, N/M \in \square} \mu(\sqrt{N/M}) H(-4M) + \delta_{k,2},$$

$H(D)$ - Hurwitz class number (weighted count of pos. def. BQFs/$\sim$)

Root number bias: proof

$$\mathrm{tr}_{S_k(N)} W_N = (-1)^{k/2} \frac{1}{2} \sum_{M|N, N/M \in \square} \mu(\sqrt{N/M}) H(-4M) + \delta_{k,2}$$

Proof. Compute $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} W_N$ via

- subtract off contribution from oldforms (nothing to do if N is squarefree!)
- $\rightsquigarrow$ alternating sum of alternating sums
- use class number relations to get positive multiple of $h_{\mathbb{Q}(\sqrt{-N})}$, apart from some exceptional cases when $N = M^2$, M squarefree

Remarks on exceptional cases:

1. $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} W_N < 0$ in exceptional cases comes from subtracting off the oldform contribution from $S_k(1)$
2. Half of exceptional cases disappear if you exclude twists from level 1 forms (when $\# \{p \mid N : p \equiv 3 \mod 4\}$ is odd)

Why a bias towards root number $+1$?

Handwaving with the adelic trace formula when $k = 2$:

$$\underset{(h \text{ test function})}{\mathrm{tr} R(h)} = \underset{\text{geom. orbital integrals}}{\sum_\gamma I_\gamma(h)} = \underset{\text{spectral distributions}}{\sum_\pi J_\pi(h)}$$

- Locally, $-W_N$ is a "positive" double coset operator $K_p \begin{pmatrix} & 1 \\ p^e & \end{pmatrix} K_p$
- $\mathrm{tr} R(h) = \mathrm{tr}_{M_k(N)}(-W_N)$ for $h \approx$ char. fun. of some set (Can actually take h to be char. fun. on def. quat. alg., at least if $v_p(N) = 1$ for some p)
- Hence geometric side is positive
- Expect most of contribution to $\mathrm{tr} R(h)$ comes from $S_k^{\mathrm{new}}(N)$

A bias for Hecke eigenvalues

Farmer–Koutsoliotas (2016): a_2 starts off negative (with positive root number, N small)

Theorem 3 (M–Pharis (2022))

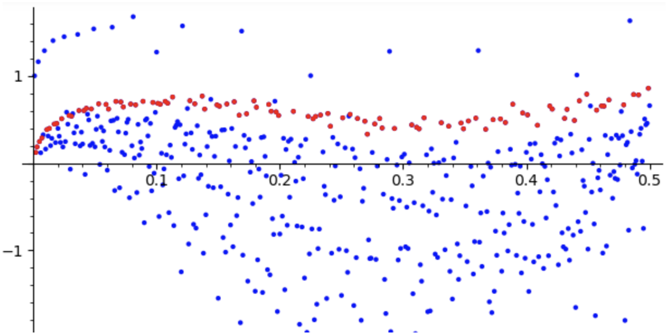
*N squarefree, $(p, N) = 1$, $N \gg p$. Then**

$$\mathrm{tr}_{S_k^{\mathrm{new}}(N)^{\pm}} T_p \sim \pm \frac{1}{4} p^{(k-2)/2} H(4pN).$$

— use Yamauchi/Skoruppa–Zagier trace formula
— point: trace formula for $T_p W_N$ is similar to that for $T_1 W_N$ when $p \ll N$
— a_p tends to be *positive* for positive root number (N large)
— N squarefree can be relaxed

*Errata at: `https://math.ou.edu/~kmartin/papers/aprank-err.pdf`

Murmurations, following [HLOP]



Averages of a_ℓ's on $S_2^{\text{new}}(N)^+$ as function of $\ell/2000$, where

— N prime, $1000 < N < 2000$

— $1 \leq \ell < 1000$ (composite or prime), $(N, \ell) = 1$

- Averages initially positive because of bias for $\ell \ll N$

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*Disclaimer for all figures: code has not been thoroughly tested

Biases and murmurations

- Biases in dimension formulas for trace formulas indicative of biases in $\mathrm{tr}\, T_\ell$'s for $\ell \ll N$
- Biases in $\mathrm{tr}\, T_\ell$'s for $\ell \ll N$ harbinger for murmurations

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Source of bias (and murmurations?) in trace formula when N squarefree, $(\ell, N) = 1$, ℓ prime:

$$\mathrm{tr}_{S_k(N)} W_N T_\ell = -\frac{1}{2}(-\ell)^{\frac{k-2}{2}} H(-4\ell N) \; + \; \delta_{k,2}\sigma_1(\ell) \qquad (N > 4\ell)$$

$$\mathrm{tr}_{S_k(N)} T_\ell = -\frac{1}{2} \sum_{s^2 \le 4\ell} p_k(s, \ell) \sum_{t \mid N} H_t(s^2 - 4\ell M') - d(N) + \delta_{k,2}\sigma_1(\ell).$$

Local biases?

$N = p_1 \dots p_r$ - squarefree.
$w_f = (-1)^{k/2} w_{p_1}(f) \cdots w_{p_r}(f)$
— $w_p(f)$ - p-th AL eigenvalue of f
— $p \parallel N \implies w_p(f)$ determines
(i) whether $a_p > 0$, and
(ii) local ramified representation $\pi_p(f)$ (Steinberg or unram quad twist)

Theorem 4 (M. 2018)

1. *There is a bias to/against* $w_{p_1} = \cdots = w_{p_t} = -1$ *based on parity of* $r + \frac{k}{2}$
2. *For fixed primes* $p_1, \dots, p_t$, *have perfect equidistribution among AL-eigenspaces under congruence conditions on* $p \mid \frac{N}{M}$, *where* $M = p_1 \dots p_t$

Ex: $t = 2$: look at eigenspaces for signs $++$, $+-$, $-+$, $--$

Where are the biases?

Can look for global biases and local biases

Significant biases for dimensions and a_ℓ's:

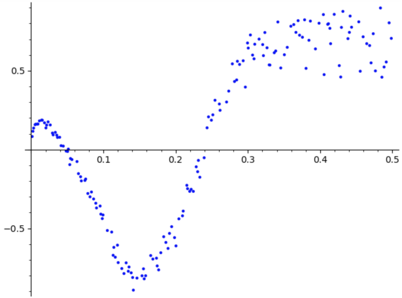
✓ Root number

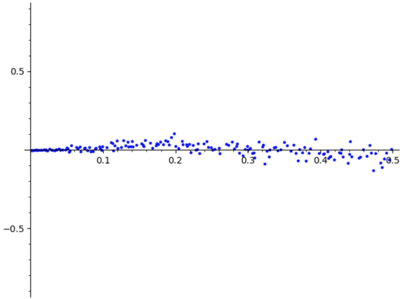
- W_M-sign, $M \mid N$, $M \gg \ell$,
  e.g., π_p - Steinberg vs unramified twist, $p \gg \ell$ ($M = p \parallel N$)
- Certain classes of π_p ramified dihedral s.c. ($p^{2j+1} \parallel N$) (Knightly...)
- L-value weighted averages
  (Michel–Ramakrishnan 2012, Feigon–Whitehouse 2009, M. 2022)

Perfect equidistribution/little bias for dimension:

- Varying Atkin–Lehner signs for small $p \parallel N$ (with cong.)
- Root number for p-minimal forms, $p^{2j} \parallel N$ (with cong.),
  i.e., π_p - unramified dihedral supercuspidal

Averages of T_ℓ's in $S_2^{\mathrm{new}}(2p)$ on AL subspace



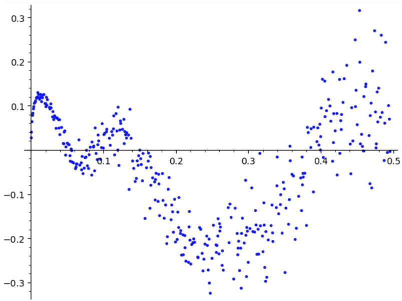


W_p+ subspace $\qquad$ W_2+ subspace

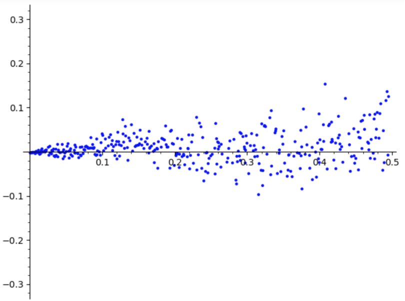
— $N = 2p$, $2000 < N < 4000$

— $\ell < 2000$ prime, $(N, \ell) = 1$

Averages of T_ℓ's in $S_2^{\mathrm{new}}(5p)$ on AL subspace



W_p+ subspace



W_5+ subspace

— $N = 5p$, $5000 < N < 10000$

— $\ell < 5000$ prime, $(N, \ell) = 1$

Where are the biases?

Can look for global biases and local biases

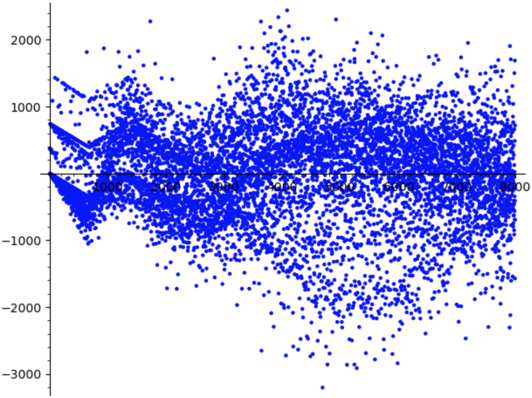
Significant biases for dimensions and a_ℓ's:

- ✓ Root number
- ✓ W_M-sign, $M \mid N$, $M \gg \ell$,
  e.g., π_p - Steinberg vs unramified twist, $p \gg \ell$ $(M = p \parallel N)$
- Certain classes of π_p ramified dihedral s.c. $(p^{2j+1} \parallel N)$ (Knightly...)
- L-value weighted averages
  (Michel–Ramakrishnan 2012, Feigon–Whitehouse 2009, M. 2022)

Perfect equidistribution/little bias for dimension:

- ✓ Varying Atkin–Lehner signs for small $p \parallel N$ (with cong.)
- Root number for p-minimal forms, $p^{2j} \parallel N$ (with cong.),
  i.e., π_p - unramified dihedral supercuspidal

L-value weighted averages of a_ℓ's

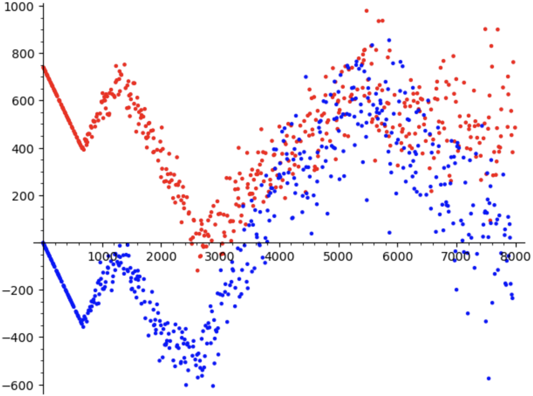


Sums of $\frac{2\sqrt{D}u_D^2}{4\pi} \times \frac{L(1,f)L(1,f,\chi_D)}{\langle f,f \rangle} a_\ell(f)$ over $S_2(N)$, where

— $D = -3$, N prime inert in $K = \mathbb{Q}(\sqrt{D})$, $2000 < N < 4000$

— $\ell < 8000$

L-value weighted averages of a_p's



Sums of $\frac{2\sqrt{D}u_D^2}{4\pi} \times \frac{L(1,f)L(1,f,\chi_D)}{\langle f,f \rangle} a_p(f)$ where

— $D = -3$, N prime inert in $K = \mathbb{Q}(\sqrt{D})$, $2000 < N < 4000$

— $p < 8000$ prime, split and inert in K

Michel–Ramakrishnan exact average when $k = 2, \Psi = 1$

$D \equiv 3 \mod 4$, $K = \mathbb{Q}(\sqrt{-D})$
$h_D = h_K$, $u_D = \frac{1}{2}|\mathfrak{o}_K^\times|$
N - prime inert in K

$$\frac{2\sqrt{D}u_D^2}{4\pi} \sum_{f \in S_2(N) \text{ new}} \frac{L(1,f)L(1,f,\chi_D)}{\langle f, f\rangle} a_\ell(f) =$$

$$\frac{12h_D^2}{N-1}\sigma_N(\ell) + u_D r(\ell D) h_D + u_D^2 \sum_{n=1}^{\ell D/N} \Phi(n, N)$$

— $\sigma_N(\ell) = \sum_{d|\ell,\,(d,N)=1} d$
— $r(\ell D)$ = number of ideals of norm ℓD in $\mathfrak{o}_K$
— $\Phi(n, N)$ - involves ideal counts and Legendre polynomial