Counting abelian surfaces with RM

Kimball Martin

The University of Oklahoma
*Visiting MIT/Harvard in Spring 2022

April 10, 2022

(joint work with Alex Cowan, Harvard)

Elliptic curves and modularity

k - field

E/k - **elliptic curve**: curve that is an (abelian) group $(E(k), O, +)$

- smooth planar cubic curve:* $y^2 = x^3 + ax + b$, O - point at infinity
- smooth projective curve of genus 1, O - some distinguished point
- if $k = \mathbb{C}$: $E = \mathbb{C}/\Lambda$, $O = 0$

Theorem (Modularity, WTWBCDT)

For any elliptic curve $E/\mathbb{Q}$, there exists a weight 2 rational newform $f = \sum a_n q^n$ such that $L(s, E) = L(s, f)$.

The map $\{E\}$ /isomorphism $\to \{f\}$ is many-to-1

The map $\{E\}$ /isogeny $\to \{f\}$ is a bijection

*if char $k \neq 2, 3$

Enumerating elliptic curves over $\mathbb{Q}$

Question

How to enumerate elliptic curves $E/\mathbb{Q}$ up to a given bound?

$E : y^2 = x^3 + ax + b$
How to order?

- Order by coefficient height: $|a| \leq A$, $|b| \leq B$ (a, b not unique for E)
- Order by **discriminant**: $\Delta = -16(4a^3 + 27b^2)$ (Δ not unique for E)
- Order by **conductor** $N \in \mathbb{N}$ (unique)

Cremona's algorithm to enumerate with $N < X$:

1. Enumerate rational newforms f in $S_2(N)$ for $N < X$
2. Compute periods of f to construct an $E \leftrightarrow f$ (cond $(E) = N$)
3. Find all $E' \sim E$ ($\sim$: isogenous)

Asymptotic counts of elliptic curves

Question

How many elliptic curves $E/\mathbb{Q}$ are there (up to isomorphism)?

E has a unique reduced minimal model over $\mathbb{Z}$:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_1, a_3, \pm a_2 \in \{0,1\}$$

$\Delta(E) \approx -64a_4^3 - 432a_6^2$

- $|\Delta(E)| < cX$ if $a_4 < X^{1/3}$ and $a_6 < X^{1/2}$.
- (at least) about $X^{1/3}X^{1/2} = X^{5/6}$ choices for (a_4, a_6) give $|\Delta(E)| < cX$

Conjecture (Brumer–McGuiness, Watkins)

The number of elliptic curves/isomorphism with (min. integral) Δ or $N < X$ is $\sim cX^{5/6}$.

Agrees well with numerics

Beyond elliptic curves

Modularity expectations:

abelian varieties	automorphic	forms	
elliptic curves	GL(2)		
abelian surfaces	GL(2)	GSp(4)	
...	...	...	...

We'll restrict to (P)GL(2) type:
$X_0(N) = \Gamma_0(N)\backslash\overline{\mathfrak{H}}$ - modular curve

Question 1

How does $J_0(N) = \mathrm{Jac}(X_0(N))$ decompose into simple abelian varieties?

The answer for varying N will tell us about counting elliptic curves, abelian surfaces of PGL(2)-type, ...

Eichler–Shimura theory

$f \in S_2(N)$ newform $\rightsquigarrow$ simple abelian variety $A_f = J_0(N)/I_f J_0(N)$

- $A_f \sim A_g \iff f = g^\sigma$ (for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)
- $\dim A_f = [K_f : \mathbb{Q}]$, $\quad K_f = \mathbb{Q}(\{a_n(f)\})$ rationality field
- $\mathrm{End}^0(A_f) = K_f$ (abelian variety is of GL(2)-type)
- cond$(A_f) = N^{\dim A_f}$

$$J_0(N) \sim J_0^{\mathrm{old}}(N) \oplus J_0^{\mathrm{new}}(N)$$

$$J_0^{\mathrm{new}}(N) \sim \bigoplus_{\{\text{newforms } f\}/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} A_f$$

- Understanding simple factors of (new part of) $J_0(N) \leftrightarrow$ understanding Galois orbits of newforms in $S_2(N)$

An example

- $S_2(35) = S_2(35)^{\text{new}}$ has 3 newforms:
  $f(z) = q + q^3 - 2q^4 - q^5 + q^7 + \dots,$
  $g_1(z) = q - \frac{1+\sqrt{17}}{2}q^2 - \frac{1-\sqrt{17}}{2}q^3 + \frac{5+\sqrt{17}}{2}q^4 + q^5 + \dots,$
  $g_2(z) = q - \frac{1-\sqrt{17}}{2}q^2 - \frac{1+\sqrt{17}}{2}q^3 + \frac{5-\sqrt{17}}{2}q^4 + q^5 + \dots$
  g_1, g_2 Galois conjugate $\rightsquigarrow$ 2 orbits: $\{f\}$ and $\{g_1, g_2\}$
- $f \leftrightarrow$ elliptic curve $E_{35a1} : y^2 + y = x^3 + x^2 - 131x - 650$
  $A_f \sim E_{35a1}$
- $g_1, g_2 \leftrightarrow$ genus 2 curve (with RM 17)
  $C : y^2 = x^6 + 2x^5 + x^4 + 8x^3 + 4x^2 + 4x + 8$
  $A_{g_1} \sim A_{g_2} \sim \mathrm{Jac}(C)$ (abelian surface of condutor 35^2)
- $J_0(35) = J_0(35)^{\text{new}} \sim E_{35a1} \oplus \mathrm{Jac}(C)$

Galois orbits of newforms

Question 1': How does $J_0^{\mathrm{new}}(N)$ *decompose into simple abelian varieties?*

is equivalent to

Question 2

How does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *act the set of newforms* $\{f_1, \ldots, f_m\}$ *in* $S_2(N)$*?*

— sizes of Galois orbits tells us degrees of K_f's [hard]
— exact action tells us rationality fields $K_f = \mathbb{Q}((a_n))$ [very hard!]

Conjecture 1 (M, 2021)

$N = p_1 \ldots p_r$ *squarefree. 100% of the time (*r *fixed,* $N \to \infty$*)* $S_2^{\mathrm{new}}(N)$ *has* 2^r *Galois orbits, i.e.,* $J_0^{\mathrm{new}}(N)$ *has* 2^r *simple factors.*

Counting small orbits

For 100% of N, expect no small orbits/abelian varieties...

Question 3

Fix $d \geq 1$. How many Galois orbits of size d in $S_2^{\mathrm{new}}(N)$, i.e., simple $J_0^{\mathrm{new}}(N)$ factors of dimension d, are there for $N < X$.

- Brumer–McGuiness/Watkins - $O(X^{5/6})$ for $d = 1$

Heuristics + data for prime $N \rightsquigarrow$

Conjecture 2 (Cowan–M)

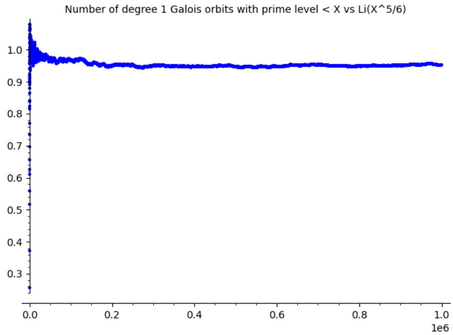
The number of Galois orbits of a fixed size d in $S_2^{\mathrm{new}}(N)$, for $N < X$ squarefree, is $O(X^{1-\frac{d}{6}+\varepsilon})$. In particular if $d \geq 7$, it is finite.

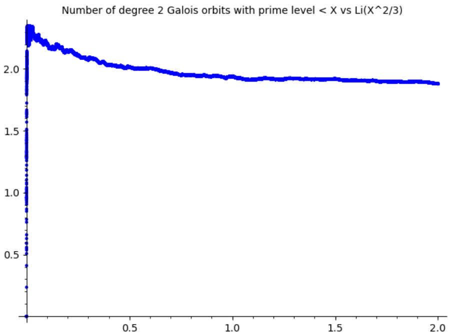
Random Hecke polynomial model

For a newform f, typically $K_f = \mathbb{Q}(a_p)$ for any p
(K. Murty (1999), Koo–Stein–Wiese (2008))

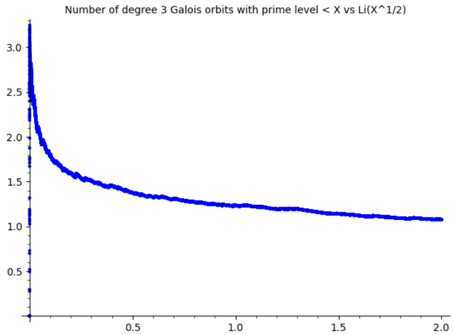
- Model the number and sizes of Galois orbits of newforms in $S_k(N)$ by the factorization type of the characteristic polynomial of T_p on $S_k(N)$
- Model the characteristic polynomial of T_p on each AL-eigenspace of dimension n as a random element of any of the following sets:
  — deg n monic polynomials in $\mathbb{Z}[x]$ with roots $\leq 2p^{\frac{k-1}{2}}$
  — Weil q-polynomials of degree $2n$, $q = p^{\frac{k}{2}}$
  — isogeny classes of n-dimensional abelian varieties over $\mathbb{F}_q$
- $h(n) = h_{k,p}(n)$ - size of above set
  Probablity of degree d factor is $\approx \frac{h(n-d)}{h(n)}$
  — precise asymptotics for $h(n)$ as $n \to \infty$ are hard
  — fit model elliptic curve counts: $\frac{h(n-1)}{h(n)} \approx cn^{-1/6}$
  $\rightsquigarrow \frac{h(n-d)}{h(n)} \approx c^d n^{-d/6}$

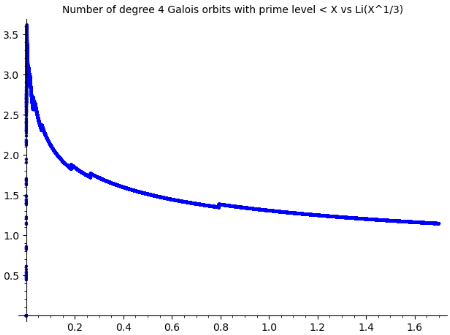
Prime level data, degrees 1 and 2





Prime level data, degrees 3 and 4





$d = 2$: a closer look

Question: Is $X^{2/3}$ the right asymptotic in Conjecture 2 for $d = 2$?

For $d = 1$, get $X^{5/6}$ by using minimal integral equations for elliptic curves.

For $d = 2$, want to estimate number of abelian surfaces/genus 2 curves with real multiplication (RM). Hard to understand RM from Weierstrass equations $y^2 = f(x)$.

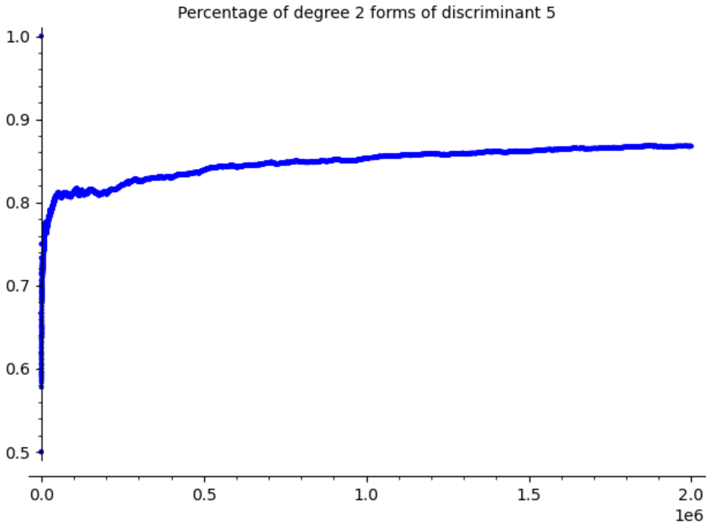
Conjecture 3 (Cowan–M)

For N squarefree, 100% of the size 2 Galois orbits $\{f, f^\sigma\}$ have rationality field $K_f = \mathbb{Q}(\sqrt{5})$.

Equivalently, 100% of abelian surfaces $A/\mathbb{Q}$ with RM (at least with conductor N^2, N squarefree) have RM 5 (i.e., $\mathbb{Z}[\frac{1+\sqrt{5}}{2}] \subset \mathrm{End}_{\mathbb{Q}}(A)$).

$\therefore$ in Conjecture 2 for $d = 2$, can restrict to RM 5

Degree 2, discriminant 5 data



Elkies–Kumar model for $Y_-(5)$

Hilbert modular surface $Y_-(5)$ parametrizes PPASs with RM 5:

$$z^2 = 2(-972g^5 - 324g^4 - 27g^3 - 4500g^2h - 1350gh + 6250h^2 - 108h)$$

birational to $\mathbb{P}^2_{m,n}$ via

$$\begin{aligned} g &= -\frac{1}{30}(-m^2 + 5n^2 + 9) \\ h &= \frac{1}{12500}(m^5 - 10m^3n^2 + 25mn^4 + 5m^4 - 50m^2n^2 + 125n^4 \\ &\quad - 5m^3 + 25mn^2 - 45m^2 + 225n^2 + 108) \end{aligned}$$

Igusa–Clebsch invariants in $\mathbb{P}^4_{(1,2,3,5)}$:

$$(I_2 : I_4 : I_6 : I_{10}) = \left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$$

Parametrizing rational curves/PPASs

$C : y^2 = f(x)$ - rational genus 2 curve with RM 5
$\rightsquigarrow A = \mathrm{Jac}(C)$ rational PPAS with RM 5
$\rightsquigarrow$ rational (z, g, h) or (m, n) on $Y_-(5)$

Question: When does a rational (z, g, h) on $Y_-(5)$ correspond to rational C, i.e., C has IC-invariants
$(I_2 : I_4 : I_6 : I_{10}) = \left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$?

Answer: $\iff h \neq 0$ and the *Mestre obstruction* vanishes.

Mestre obstruction: Mestre conic $L = L(I_2, I_4, I_6, I_{10})$ needs to have a rational point

Mestre conic

$$
\begin{aligned}
L: &-189843750\left(96\,g^3+337\,g^2+108\,g-400\,h+9\right)x^2\\
&+5062500\left(144\,g^4+1299\,g^3+754\,g^2-2000\,gh+144\,g-500\,h+9\right)xy\\
&-3750\,(1944\,g^5+40905\,g^4+36990\,g^3-68400\,g^2h+11835\,g^2-43200\,gh\\
&\qquad+50000\,h^2+1620\,g-5400\,h+81)y^2\\
&-7500\,(1944\,g^5+40905\,g^4+36990\,g^3-68400\,g^2h+11835\,g^2-43200\,gh\\
&\qquad+50000\,h^2+1620\,g-5400\,h+81)xz\\
&+900\,(324\,g^6+14931\,g^5+19395\,g^4-25800\,g^3h+9105\,g^3-30100\,g^2h\\
&\qquad+2020\,g^2-8400\,gh+10000\,h^2+216\,g-700\,h+9)yz\\
&-(2916\,g^7+283338\,g^6+499041\,g^5-496800\,g^4h+319140\,g^4\\
&-915300\,g^3h+525000\,g^2h^2+101160\,g^3-426300\,g^2h+500000\,gh^2+17214\,g^2\\
&\qquad-76800\,gh+100000\,h^2+1512\,g-4800\,h+54)z^2
\end{aligned}
$$

$$\text{Disc}: \quad 2^6\cdot 3^3\cdot 5^{22}\cdot h^2(-9g^2+8h)^2z^2$$

Generic criterion

Theorem 1 (Cowan–M)

Generically, the rational (z, g, h) *or* (m, n)*'s on* $Y_{-}(5)$ *corresponding to rational genus 2* C *with (rational) RM 5 are those satisfying* $30g + 4 = m^2 - 5n^2 - 5 \in \mathbb{Q}$ *is a norm from* $\mathbb{Q}(\sqrt{5})$

Proof.

1. After calculating many curves of small height, guess that the Mestre obstruction only depends on g.
2. After more calculations, guess the Mestre conic is equivalent to $x^2 - 5y^2 - (30g + 4)z^2$. (It's not over $\mathbb{Q}(g, h)$!)
3. Spend months trying to reduce the Mestre conic.

□

Future goals: use this to estimate counts of PPAS/genus 2 curves with RM by discriminant and conductor.

PS: I_{10} invariants for rational Hilbert modular surfaces

$D = 5$: $(262144) \cdot (m^5 - 10m^3n^2 + 25mn^4 + 5m^4 - 50m^2n^2 + 125n^4 - 5m^3 + 25mn^2 - 45m^2 + 225n^2 + 108)^2$

$D = 8$: $(-\frac{1}{131072}) \cdot (m+1) \cdot (m-1)^3 \cdot (2n^2-1)^{-5} \cdot (-16m^2n^2 + 32n^4 + m^3 - 56mn^2 + 9m^2 - 72n^2 + 27m + 27)^2$

$D = 12$: $(-32) \cdot (n+1)^3 \cdot (n-1)^4 \cdot (-m^2 + 27n^2 - 27)^{-3} \cdot (mn^2 + 9n^2 - 8)^3$

$D = 13$:
$(\frac{256}{4782969}) \cdot (-12m^3 + 3m^2 + n^2)^{-6} \cdot (267m^3 - 72m^2n + mn^2 + 3552m^2 - 1440mn + 128n^2 - 768m)^2 \cdot (-m^3 - 150m^2 + 6mn - 264m + 120n + 64)^4$

$D = 17$:
$(\frac{1024}{4782969}) \cdot (-132m + n + 3)^3 \cdot (456m^2 + mn + 723m - 8n + 24)^3 \cdot (4608m^3 - 1728m^2 + n^2 + 216m - 9)^{-8} \cdot (-256m^3 - 1200m^2 + 18mn - 6006m + 99n + 41)^5$