Rep Thy I: Problem Set 1 (due Fri Aug 31)

1. (a) Show C_n has a faithful 1-dimensional complex representation for all n.

   (b) Show C_n has a faithful n-dimensional rational representation ($\rho : C_n \to \mathrm{GL}_n(\mathbb{Q})$) for all n.

   (c) Show C_4 has a faithful 2-dimensional rational representation.

2. Let G be a finite group and ρ a representation of G. Show $\rho(g)$ is diagonalizable for all $g \in G$.

3. Let A be a finite abelian group.

   (a) Show any irreducible representation of A is 1-dimensional.

   (b) Show that if A is noncyclic, then any 1-dimensional representation of A is non-faithful.

   (c) Describe the smallest n (in terms of invariants of A) for which you can construct a faithful n-dimensional (complex) representation.

4. Let $G = D_{2n}$. Construct a faithful 2-dimensional representation of G (you may specify it by writing down matrices that generators map to). Is this representation irreducible?

5. Let $G = C_2 \times C_2$ and ρ be the regular representation of G. Determine the irreducible subrepresentations of G, specifying their dimensions and which are isomorphic to each other.

6. Suppose $V = \mathbb{C}^2$ and write an operator $\rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in matrix form with respect to the standard basis $\{e_1, e_2\}$. Compute $\mathrm{Sym}^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \mathrm{Sym}^2(\rho)(g)$ and $\Lambda^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \Lambda^2(\rho)(g)$ as matrices (with respect to suitable bases).

7. Let (ρ, V) be a representation of G. Prove $\mathrm{Sym}^2(V)$ is stable in $V \otimes V$.

Presentations

EC (1), JD (3), DG (4), EH (5), JL (6)