**LANGLANDS CONJECTURE IN THE TETRAHEDRAL AND OCTAHEDRAL CASES**

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These are notes on Langlands' proof of his reciprocity conjecture in the tetrahedral case and Tunnell's proof in the octahedral case based on Gelbart's exposition [Ge] and Tunnell's paper [Tu]. These notes were written under the guidance of Dinakar Ramakrishnan.

## 1. The Artin and Langlands Conjectures

Let $F$ be a number field, $K$ a finite Galois extension and $\sigma$ a Galois representation $\sigma : \mathrm{Gal}(K/F) \to \mathrm{GL}_n(\mathbb{C})$. Recall the Artin $L$-function is defined on some right-half plane by $L(s,\sigma) = \prod_v L(s,\sigma_v)$ where the product is over all places $v$ of $F$ and if $v$ is unramified

$$L(s,\sigma_v) = \frac{1}{det(1-\sigma_v(\mathrm{Fr}_v)q_v^{-s})}.$$

Here $\mathrm{Fr}_v$ is the Frobenius and $q_v$ is the size of the residue field $k_v$. For a complete definition of Artin $L$-functions, see [Ne] or [Ma].

**Conjecture 1.** *(Artin) If $\sigma$ is irreducible and nontrivial, then $L(s,\sigma)$ extends to an entire function. Equivalently, if $\sigma$ (not necessarily irreducible) does not contain the trivial representation, then $L(s,\sigma)$ extends to an entire function.*

Artin proved the conjecture in the case $\sigma$ is *monomial*, i.e. induced from a degree one representation of some subgroup. Brauer showed for any $\sigma$, $L(s,\sigma)$ extends to a meromorphic function on all of $\mathbb{C}$.

It has also been proven for several two-dimensional $\sigma$. Since conjugation of matrices preserves trace, $\mathrm{GL}_2(\mathbb{C})$ acts by conjugation on the three-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, the $2\times 2$ matrices of trace zero. Let us take

$$\mathbf{r} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

as our basis for $\mathfrak{sl}_2(\mathbb{C})$. Then conjugation by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $w = ad-bc$ sends $\mathbf{r} \to (a^2\mathbf{r} - ac\mathbf{s} - c^2\mathbf{t})/w$, $\mathbf{s} \to \mathbf{s}$ and $\mathbf{t} \to (-b^2\mathbf{r} + bd\mathbf{s} + d^2\mathbf{t})/w$. So this representation, the *adjoint representation*, $Ad : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$, is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 0 & -b^2 \\ -ac & ad-bc & bc \\ -c^2 & 0 & d^2 \end{pmatrix} w^{-1}.$$

Note that the kernel of $Ad$ is just the scalar matrices and it acts on diagonal matrices by

$$Ad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \frac{a}{b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{b}{a} \end{pmatrix}.$$

Therefore $Ad$ factors through $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* = \mathrm{PGL}_2(\mathbb{C})$ and thus the image of $Ad$ lies inside a subgroup of $\mathrm{GL}_3(\mathbb{C})$ isomorphic to $\mathrm{PGL}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{C})$.

So the image of $Ad\circ\sigma$ is a finite subgroup (the Galois group is finite) of $\mathrm{SO}_3(\mathbb{C})$ and we classify $\sigma$ according to the type of this subgroup. The finite subgroups of $\mathrm{SO}_3(\mathbb{C})$ (and thus two-dimensional $\sigma$) fall into five categories:

(1) $Ad\circ\sigma$ is cyclic $\implies$ $\sigma$ is reducible or monomial
(2) $Ad\circ\sigma$ is dihedral $\implies$ $\sigma$ is monomial
(3) $Ad\circ\sigma$ is tetrahedral, i.e. isomorphic to $A_4$
(4) $Ad\circ\sigma$ is octahedral, i.e. isomorphic to $S_4$

(5) $Ad \circ \sigma$ is icosahedral, i.e. isomorphic to $A_5$

Accordingly we call $\sigma$ *cyclic, dihedral, tetrahedral, octahedral* or *icosahedral.* A proof that the irreducible cyclic and dihedral representations must be monomial is given in [Ro] and a proof that all dihedral representations are irreducible is given in [Se]. So the irreducible cyclic and dihedral cases were known to Artin. If $\sigma$ is tetrahedral, octahedral or icosahedral, then $\sigma$ must be irreducible as it is two-dimensional and its image is nonabelian. Langlands proved the tetrahedral case [La] and Tunnell proved the octahedral case [Tu] by proving the more general Langlands conjecture in those cases. The icosahedral case has not yet been settled for either the Artin or the Langlands conjecture. (The Artin and Langlands conjecture are actually equivalent for $n = 2$ or 3.)

**Theorem 1.** *Let $\pi$ be an irreducible automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then we can associate Euler factors $L(s, \pi_v)$ to each $v$ and define a holomorphic $L$-function $L(s,\pi) = \prod L(s, \pi_v)$ in some right-half plane. Moreover $L(s,\pi)$ satisfies a functional equation and extends to an meromorphic function on $\mathbb{C}$, which is actually entire if $\pi$ is cuspidal and nontrivial.*

For more information on automorphic forms, representations and their $L$-functions, see [Ge3], [Kn], or [Ge2]. Now we are ready to state the Langlands conjecture,

**Conjecture 2.** *(Langlands reciprocity) Let $K/F$ be a (finite) Galois extension of number fields. To each Galois representation $\sigma : \mathrm{Gal}(K/F) \to \mathrm{GL}_n(\mathbb{C})$, we can associate an automorphic representation $\pi(\sigma)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ so that $L(s, \sigma_v) = L(s, \pi_v)$ for almost all $v$. Further, if $\sigma$ is nontrivial and irreducible, then $\pi(\sigma)$ is cuspidal.*

If there is such an automorphic representation corresponding to a given $\sigma$, we will say $\pi(\sigma)$ *exists.* If $\pi(\sigma)$ exists, it will be unique by the so-called Strong Multiplicity One Theorem. But in our case, uniqueness will come naturally out of the proofs. The "irreducible $\implies$ cuspidal" part is known and stated below in Corollary 1. Langlands' conjecture implies Artin's Conjecture by Theorem 1 and the following,

**Theorem 2.** *If $L(s, \pi_v) = L(s, \sigma_v)$ for almost all $v$, then in fact $L(s, \pi) = L(s, \sigma)$.*

In the classical case where $F = \mathbb{Q}$ and $\sigma$ is an irreducible two-dimensional $\mathbb{C}$-representation of $K$ sending complex conjugation to an element of determinant -1, Langlands reciprocity asserts that we can associate a classical cusp form $\pi(\sigma)$ of weight 1 to $\sigma$ such that their $L$-functions coincide almost everywhere. In this case every classical cusp form of weight 1 comes from such an irreducible two-dimensional $\mathbb{C}$-representation. However in general, not all automorphic representations will come from complex Galois representations.

Recently (1998, Harris-Taylor, Henniart) the Local Langlands Conjecture was proven over $p$-adic fields, i.e. finite extensions of $\mathbb{Q}_p$. (The result for archimedean local fields was known before this.) We will state this result, but only use the $\mathrm{GL}_2$ case here which was proved by Jaquet-Langlands, Tunnell and Kutzko in 1980. The local case is somewhat different from the global case, as it involves the representations of the Weil group (a dense subgroup of the Galois group) or the extended Weil group instead of Galois representations, and the correspondence is actually a bijection. The correspondence is a bijection preserving $L$- and $\epsilon$- factors laid out as follows

$$\begin{array}{ccc}
 & & \{\text{irreducible admissible representations of } \mathrm{GL}_n(F)\} \\
 & & || \\
\{n-\text{dimensional representations} & & \{\text{principal series, special and} \\
\text{of the extended Weil group of } F\} & \longleftrightarrow & \text{supercuspidal representations of } \mathrm{GL}_n(F)\} \\
\bigcup & & \bigcup \\
\{n-\text{dimensional representations} & & \{\text{principal series and} \\
\text{of the Weil group of } F\} & \longleftrightarrow & \text{supercuspidal representations of } \mathrm{GL}_n(F)\} \\
\bigcup & & \bigcup \\
\{\text{irreducible } n-\text{dimensional} & & \{\text{supercuspidal representations} \\
\text{representations of the Weil group of } F\} & \longleftrightarrow & \text{of } \mathrm{GL}_n(F)\}
\end{array}$$

A precise statement for the latter correspondence is given below in Theorem 3. To prove these correspondences, it suffices to prove it between the bottom two sets. For a $p$-adic field $F$ with residue field $k$ and $q = |k|$, the Weil group $W_F$ is defined to be the set of $\sigma$ in the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ such that $\sigma$ induces an automorphism of the form $x \to x^{q^n}$ on $\bar{k}$ for some $n$. Here the extended Weil group is

$W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$. Weil groups actually can be associated to all local and global fields. Introductions to Weil groups can be found in [Kn] and [Ta].

**Theorem 3.** *(Local Langlands) Let $F$ be a local field. There is a unique bijection between irreducible $n$-dimensional representations $\sigma$ of the Weil group $W_F$ and supercuspidal representations $\pi$ of $\mathrm{GL}_n(F)$, $\sigma \to \pi(\sigma)$, that preserves $L$- and $\epsilon$- factors of all the twists*

$$L(s, \sigma \otimes \tau) = L(s, \pi(\sigma) \otimes \pi(\tau)) \ , \ \ \epsilon(s, \sigma \otimes \tau) = \epsilon(s, \pi(\sigma) \otimes \pi(\tau)),$$

*for all irreducible representations $\tau$ of $W_F$ with $\dim \tau \leq \max\{1, \dim \sigma - 2)\}$. In particular, $\det \sigma = \omega_{\pi(\sigma)}$, the central character of $\pi(\sigma)$. Moreover $\sigma$ is unramified if and only if $\pi(\sigma)$ is unramified.*

For a elementary overview of the Local Langlands Conjecture, see [Ro2]. See [Ta] for a formulation of the Local Langlands Correspondence in terms of the so-called Weil-Deligne groups.

## 2. Automorphic forms on $\mathrm{GL}_2$ and $\mathrm{GL}_3$

First we introduce the notion of base change. Let $E/F$ be a Galois extension of number fields, $v$ a place of $F$ and $w$ a place of $E$ above $v$. Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi_v'$ be cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_E)$, respectively. Suppose $\pi_v$ is unramified (recall it is for almost all $v$). By Local Langlands, $\pi_v = \pi_v(\sigma_v)$ for some unramified representation $\sigma_v$ of the Weil group $W_{F_v}$. The Weil group $W_{E_w}$ equals $W_{F_v} \cap \mathrm{Gal}(\bar{E}/E)$, so $\sigma_v$ restricts to a representation $\Sigma_w$. We say $\pi'$ is a *base change lift* of $\pi$ if $\pi_w' = \pi_w'(\Sigma_w)$ for almost every $w$, and we write $\pi' = BC_{E/F}\pi$.

The group $\mathrm{Gal}(E/F)$ acts on $\mathrm{GL}_2(E)\backslash \mathrm{GL}_2(\mathbb{A}_E)$, and thus also on automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$. If $\pi'$ is invariant under $\mathrm{Gal}(E/F)$ we say it is *Galois invariant.*

**Theorem 4.** *Let $E/F$ be cyclic extension of number fields of prime degree. Then*

*(i) for each cuspidal representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, the base change $BC_{E/F}\pi$ to $E$ exists;*

*(ii) (descent) a cuspidal representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_E)$ equals $BC_{E/F}\pi$ for some $\pi$ if and only if $\pi'$ is Galois invariant; and*

*(iii) if $\pi$ and $\pi'$ are cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ then $BC_{E/F}\pi = BC_{E/F}\pi'$ if and only if $\pi' = \pi \otimes \omega$ for some character $\omega$ of $F^*\mathfrak{N}_{E/F}(\mathbb{A}_E^*)\backslash \mathbb{A}_F^*$.*

*Further if $K \supseteq E \supseteq F$ are any extensions (i.e., not necessarily cyclic nor prime degree) of number fields, then*

*(iv) (compatiblility with reciprocity) if $\sigma$ is a representation of $\mathrm{Gal}(K/F)$, $\sigma_E = \sigma|_{\mathrm{Gal}(K/E)}$, $\pi = \pi(\sigma)$ and $\pi(\sigma_E)$ exist, then $BC_{E/F}\pi = \pi(\sigma_E)$;*

*(v) (transitivity) $BC_{K/F}(\pi) = BC_{K/E}(BC_{E/F}(\pi))$ provided the base changes exist; and*

*(vi) (compatible with twisting) if $\chi$ is an idele class character of $F$ and $\chi_E = \chi \circ N_{E/F}$, then*

$$BC_{E/F}(\pi \otimes \chi) = BC_{E/F}(\pi) \otimes \chi_E.$$

Now we define the $\mathrm{GL}_3$-lift (also called the *adjoint-square lift* as it corresponds with the adjoint representation on the Galois side). Let $\pi$ (resp. $\Pi$) be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ (resp. $\mathrm{GL}_3(\mathbb{A}_F)$). For almost all $v$ we can write $\pi_v$ as $\pi_v(\sigma_v)$ for some $\sigma_v$. Then $\Pi$ is a $\mathrm{GL}_3$-*lift* of $\pi$ if $\Pi_v = \Pi_v(Ad \circ \sigma_v)$ for almost all $v$.

**Theorem 5.** *For every cuspidal automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, the $\mathrm{GL}_3$-lift $\Pi$ exists. Further, $\Pi$ is cuspidal if and only if $\pi$ is cuspidal and not of the form $\pi(\sigma)$ for some monomial $\sigma$.*

To prove the main theorem, we will use the $\mathrm{GL}_3$-lift along with the following result.

**Theorem 6.** *Let $\pi, \pi'$ be automorphic representations of $\mathrm{GL}_3(\mathbb{A}_F)$. Then we can define a meromorphic $L$-function $L(s, \pi \times \pi')$ on all of $\mathbb{C}$ with local factors $L(s, \pi_v \times \pi_v')$. Furthermore if $\pi'$ is the contragradient $\check{\pi}$ of $\pi$, then:*

*(i) when $\pi$ is cuspidal, $L(s, \pi \times \check{\pi})$ has a pole at $s = 1$ if and only if $\pi$ is isomorphic to $\check{\pi}$, in which case the pole is simple; and*

*(ii) $L(1, \pi_v \times \check{\pi}_v)$ is non-zero and finite at every $v$.*

Jacquet and Shalika proved the following,

**Theorem 7.** *Let $\pi$ be a automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then $L(s, \pi \times \check{\pi})$ has a simple pole at $s = 1$ if and only if $\pi$ is cuspidal.*

This compares with the result for Galois representations:

**Theorem 8.** *Let $\sigma$ be a Galois representation. Then $L(s, \sigma \times \check{\sigma})$ has a simple pole at $s = 1$ if and only if $\sigma$ is irreducible.*

As $L(s, \pi(\sigma) \times \check{\pi(\sigma)}) = L(s, \sigma \times \check{\sigma})$, the above theorems yield the following,

**Corollary 1.** *Suppose $\pi = \pi(\sigma)$. Then $\sigma$ is irreducible if and only if $\pi$ is cuspidal.*

## 3. The Tetrahedral Case

Now we will prove the tetrahedral case of Langlands reciprocity in $\mathrm{GL}_2$.

**Theorem 9.** *Let $K/F$ be a Galois extension of number fields with Galois group $G$. Let $\sigma$ be an (irreducible) two-dimensional tetrahedral representation of $G$. Then there exists a unique cuspidal automorphic representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $L(s, \sigma_v) = L(s, \pi_v)$ for almost all $v$.*

**Proof.** First we will define a representation $\varpi(\sigma)$. Then we show that actually $\varpi(\sigma) = \pi(\sigma)$, i.e. we show $L(s, \sigma_v) = L(s, \varpi(\sigma)_v)$ for almost all $v$.

*Step 1*. Define $\varpi(\sigma)$.

Recall $\sigma$ being tetrahedral means that the image of $Ad \circ \sigma$ is isomorphic to $A_4$. Since $V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $A_4$, we can let $E$ be the intermediate field $K \supset E \supset F$ that corresponds to the subgroup $H = (Ad \circ \sigma)^{-1}(V_4)$ of $G$. As the map $G \to A_4 \to \mathbb{Z}/3\mathbb{Z}$ has kernel $H$, $H$ is normal in $G$. Then $\sigma$ restricts to a representation $\Sigma$ of $H = \mathrm{Gal}(K/E)$ with $(Ad \circ \Sigma)(H) = V_4$. Thus $\Sigma$ is monomial and we already know Langlands reciprocity is true for $\Sigma$, i.e. $\pi(\Sigma)$ exists. Moreover $\Sigma$ is dihedral and therefore irreducible, so $\pi(\Sigma)$ is cuspidal.

As $H$ is normal in $G$, $G$ acts by conjugation on $H$ and thus on representations of $H$. The representation $\Sigma$ is invariant under this action, being the restriction of a representation of $G$. Then by reciprocity $\pi(\Sigma)$ is Galois invariant. Therefore by Theorem 4(ii), for some cuspidal representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_F)$, we have $\pi(\Sigma) = BC_{E/F}(\pi')$. Part (iii) of this theorem further states that all the representations whose base change to $E$ will be $\pi(\Sigma)$ are precisely the ones of the form $\pi' \otimes \omega$, where $\omega$ is a character of $F^*\mathfrak{N}_{E/F}(\mathbb{A}_E^*)\backslash\mathbb{A}_F^*$.

For a representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, we can write

$$\pi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega_\pi(a)\left(\pi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), a \in \mathbb{A}_F^*.$$

Then $\omega_\pi$ is a character of $\mathbb{A}_F^*$ called the *central character* of $\pi$.

Let $\omega_{\pi'}$ be the central character of $\pi'$. Then the central character of $\pi' \otimes \omega$ will be $\omega_{\pi'}\omega^2$. Consider the central character $\omega_{\pi(\sigma)}$ of $\pi(\sigma)$ if it exists. Then $L(s, \pi(\sigma_v)) = L(s, \sigma_v)$ a.e. $v$ implies $\omega_{\pi(\sigma),v} = \det \sigma_v$ a.e. $v$. By a Theorem of Hecke (Corollary 7-34 of [Ra]), we actually have the equality $\omega_{\pi(\sigma)} = \det \sigma$. So if $\pi' \otimes \omega = \pi(\sigma)$, it must be that $\omega_{\pi'}\omega^2 = \det \sigma$. By class field theory, $F^*\mathfrak{N}_{E/F}(\mathbb{A}_E^*)\backslash\mathbb{A}_F^* \simeq \mathrm{Gal}(E/F) \simeq \mathbb{Z}/3\mathbb{Z}$. Since $\omega^3 = 1$, choosing $\omega$ such that $\omega_{\pi'}\omega^2 = \det \sigma$ determines $\omega$ uniquely. Choose this $\omega$ and define $\varpi(\sigma)$ to be $\pi' \otimes \omega$. We have seen that if $\pi(\sigma)$ exists, it must be $\varpi(\sigma)$. So once we show $L(s, \varpi(\sigma)_v) = L(s, \sigma_v)$ for almost all $v$, we will have $\varpi(\sigma) = \pi(\sigma)$ and the uniqueness of $\varpi(\sigma)$ will imply the uniqueness of $\pi(\sigma)$.

*Step 2*. Show $L(s, \varpi(\sigma)_v) = L(s, \sigma_v)$ for almost all $v$.

By Local Langlands, we can write $\varpi(\sigma) = \otimes\pi_v$ with $\pi_v = \pi_v(\sigma'_v)$ for each unramified $\pi_v$. We will show $\sigma'_v = \sigma_v$ for almost all $v$. By our choice of $\varpi(\sigma)$ and the definition of base change we already know $\sigma'_v$ is a two-dimensional representation of $W_{F_v}$ such that $\sigma'_v = \sigma_v$ on $W_{E_w}$.

So it suffices to show $\sigma'_v = \sigma_v$ in the case where $\sigma'_v$ and $\sigma_v$ are unramified. Then they are completely reducible and we can write

$$\sigma_v(\mathrm{Fr}_v) = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} \quad , \quad \sigma'_v(\mathrm{Fr}_v) = \begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix}.$$

We need only show these matrices are conjugate as the representations $\sigma_v, \sigma'_v$ are determined by their action on the Frobenius $\mathrm{Fr}_v$. We know $\sigma_v(\mathrm{Fr}_v^3) = \sigma'_v(\mathrm{Fr}_v^3)$ as $\mathrm{Fr}_v^3 \in W_{E_w}$. Thus

$$\begin{pmatrix} a_v^3 & 0 \\ 0 & b_v^3 \end{pmatrix} \sim \begin{pmatrix} c_v^3 & 0 \\ 0 & d_v^3 \end{pmatrix}.$$

So for some cube roots of 1, $\xi$ and $\xi'$, we have $c_v = \xi a_v$ and $d_v = \xi' b_v$. But $\omega_{\varpi(\sigma)} = \det \sigma$ implies $\det \sigma'_v = \det \sigma_v$ so we get $\xi\xi' = 1$, i.e. $\xi' = \xi^2$. It remains to show $\xi = 1$. This can be easily done using the following

**Lemma.** *For almost all $v$, $Ad \circ \sigma'_v = Ad \circ \sigma_v$.*

**Proof of Lemma.** As every representation of $A_4$ is monomial, we already know $\pi_1 = \pi(Ad \circ \sigma)$ exists as a representation of $\mathrm{GL}_3(\mathbb{A}_F)$. If $\varpi(\sigma) = \pi(\sigma')$ for some $\sigma'$, then $\sigma' = \sigma$ so $\sigma'$ would not be monomial. Hence by Theorem 5, $\varpi(\sigma)$ lifts to a cuspidal representation $\pi_1^*$ of $\mathrm{GL}_3(\mathbb{A}_F)$. It suffices to show $\pi_1 = \pi_1^*$ because $(\pi_1)_v = \pi(Ad \circ \sigma_v)$ and $(\pi_1^*)_v = \pi(Ad \circ \sigma'_v)$ for almost all $v$.

Consider the $L$-functions

$$L(s, \pi_1 \times \check{\pi}_1) = \prod_v L(s, (\pi_1)_v \times (\check{\pi}_1)_v) \quad , \quad L(s, \pi_1^* \times \check{\pi}_1) = \prod_v L(s, (\pi_1^*)_v \times (\check{\pi}_1)_v),$$

where $\check{\pi}_1$ denotes the contragradient representation of $\pi_1$. Since the $\mathrm{GL}_3$-lift preserves $L$-functions, we have

$$L(s, (\pi_1)_v \times (\check{\pi}_1)_v) = L(s, (Ad \circ \sigma_v) \otimes (Ad \circ \check{\sigma}_v)) \quad , \quad L(s, (\pi_1^*)_v \times (\check{\pi}_1)_v) = L(s, (Ad \circ \sigma'_v) \otimes (Ad \circ \check{\sigma}_v))$$

for almost all $v$. Now we claim that

$$L(s, (\pi_1)_v \times (\check{\pi}_1)_v) = L(s, (\pi_1^*)_v \times (\check{\pi}_1)_v)$$

for almost all $v$.

As $Ad \circ \sigma$ factors through $A_4$, it can be identified with the irreducible three-dimensional representation of $A_4$. Thus, since the irreducible three-dimensional representation of $A_4$ is induced from characters of $V_4$. $Ad \circ \sigma$ must be induced from an idele class character $\theta$ of $E$. So

$$Ad \circ \check{\sigma}_v = \oplus_{w|v} \mathrm{Ind}_{\mathrm{W_{E_w}}}^{\mathrm{W_{F_v}}} \theta_{\mathrm{w}}.$$

This gives

$$(Ad \circ \sigma_v) \otimes (Ad \circ \check{\sigma}_v) = \oplus_{w|v} \mathrm{Ind}_{\mathrm{W_{E_w}}}^{\mathrm{W_{F_v}}} (\Sigma_{\mathrm{w}} \otimes \theta_{\mathrm{w}}), (\mathrm{Ad} \circ \sigma'_{\mathrm{v}}) \otimes (\mathrm{Ad} \circ \check{\sigma}_{\mathrm{v}}) = \oplus_{\mathrm{w|v}} \mathrm{Ind}_{\mathrm{W_{E_w}}}^{\mathrm{W_{F_v}}} (\Sigma'_{\mathrm{w}} \otimes \theta_{\mathrm{w}}),$$

where $\Sigma'$ denotes the restriction of $\sigma'$ to $H$. Then for almost all $w$, $\Sigma_w = \Sigma'_w$, so for almost all $v$,

$$\begin{aligned} L(s, (\pi_1)_v \times (\check{\pi}_1)_v) &= L(s, (Ad \circ \sigma_v) \otimes (Ad \circ \check{\sigma}_v)) \\ &= L(s, (Ad \circ \sigma'_v) \otimes (Ad \circ \check{\sigma}_v)) \\ &= L(s, (\pi_1^*)_v \times (\check{\pi}_1)_v). \end{aligned}$$

Hence for a finite set $S$ of primes,

$$L(s, \pi_1^* \times \check{\pi}_1) = \left( \prod_{v \in S} \frac{L(s, (\pi_1^*)_v \times (\check{\pi}_1)_v)}{L(s, (\pi_1)_v \times (\check{\pi}_1)_v)} \right) L(s, \pi_1 \times \check{\pi}_1).$$

By Theorem 6, the finite product is non-zero and so the right hand side (and thus the left) has a pole at $s = 1$. The same Theorem yields that in fact $\pi_1 = \pi_1^*$. $\square$

Now we finish the proof of the main theorem. Since the kernel of $Ad$ is just the scalar matrices, the Lemma says that for some $\lambda \neq 0$,

$$\begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix} = \begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \sim \begin{pmatrix} \lambda a_v & 0 \\ 0 & \lambda b_v \end{pmatrix}.$$

Recall it only remains to show that $\xi = 1$, so we need only show $\lambda = 1$. Either the latter two matrices are equal, in which case $\lambda = \xi = \xi^2 = 1$, or they are conjugate by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, in which case $\lambda a_v = \xi^2 b_v$ and

$\lambda b_v = \xi a_v$. So we may assume the latter case and the equations yield $\lambda^2 = 1$ and $b_v = \lambda \xi a_v$. If $\lambda = -1$, then

$$(Ad \circ \sigma_v)(\mathrm{Fr}_v) = \begin{pmatrix} \frac{a_v}{b_v} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{b_v}{a_v} \end{pmatrix} = \begin{pmatrix} -\xi^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\xi \end{pmatrix},$$

is an element of order 6 of $A_4 \subset \mathrm{PGL}_2(\mathbb{C})$. But $A_4$ has no elements of order 6, so we must have $\lambda = 1$. $\square$

## 4. The Octahedral Case

Here we prove the octahedral case of Langlands conjecture,

**Theorem 10.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$. Let $\sigma$ be a two-dimensional octahedral representation of $G$. Then there exists a unique automorphic representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $L(s, \sigma_v) = L(s, \pi_v)$ for almost all $v$. Further if $\sigma$ is nontrivial and irreducible, then $\pi(\sigma)$ is cuspidal.*

For this proof, we use the following result on nonnormal base change due to Jacquet, Piatetski-Shapiro and Shalika.

**Theorem 11.** *Let $K/F$ be a nonnormal cubic extension of number fields and $\pi$ a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then $BC_{K/F}(\pi)$ exists.*

**Proof of Theorem 10.**
Let $K$ and $E$ be the intermediate fields of $L/F$ corresponding to a Sylow 2-subgroup and the $A_4$-subgroup of $G$, respectively. Let $M$ be the compositum of $K$ and $E$. So $[K : F] = 3$, $[E : F] = 2$ and $[L : M] = 4$. If $N$ is an intermediate field of $L/F$, let $\sigma_N$ denote the restriction of $\sigma$ to $\mathrm{Gal}(L/N)$.

Since $\sigma_M$ and $\sigma_K$ are monomial, we already know $\pi(\sigma_M)$ and $\pi(\sigma_K)$ exist. Similarly by the tetrahedral case above, we know $\pi(\sigma_E)$ exists. Since $[E : F] = 2$ there are two cuspidal representations $\pi_1, \pi_2$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $BC_{E/F}(\pi_i) = \pi(\sigma_E)$ and $\pi_1 = \pi_2 \otimes \omega_{E/F}$.

**Lemma.** *There is a unique $i$ such that $BC_{K/F}(\pi_i) = \pi(\sigma_K)$.*

**Proof of Lemma.** By nonnormal cubic base change, $\Pi_i = BC_{K/F}(\pi_i)$ exists for $i = 1, 2$. Then Theorem 4(iv),(v) implies $BC_{M/K}(\Pi_i) = \pi(\sigma_M)$. By Theorem 4(iii), this gives that $\Pi_1 = \Pi_2 \otimes \omega_{M/K}$. If $\Pi_1 = \Pi_2$, then $\Pi_1 = \Pi_1 \otimes \omega_{M/K}$ which would imply $\pi(\sigma_M)$ is not cuspidal by a Theorem of Labesse and Langlands. However $\sigma_M$ is dihedral and therefore irreducible so $\pi(\sigma_M)$ is cuspidal (Corollary 1) and thus $\Pi_1 \neq \Pi_2$.

Since $[M : K] = 2$ there are exactly two representations $\pi$ such that $BC_{M/K}(\pi) = \pi(\sigma_M)$, they are precisely $\Pi_1$ and $\Pi_2$. But $\pi(\sigma_K)$ is also one such representation, i.e. $\pi(\sigma_K)$ equals $\Pi_1$ or $\Pi_2$ and we are done. $\square$

Let $\pi$ be the $\pi_i$ such that $BC_{K/F}(\pi_i) = \pi(\sigma_K)$. We proceed to show that $\pi = \pi(\sigma)$ (i.e., that $L(s, \pi_v) = L(s, \sigma_v)$ for almost all $v$) thus proving the theorem. Consider a place $v$ of $F$ such that $\sigma_v$ and $\pi_v$ are unramified. By Local Langlands, $\pi_v = \pi_v(\sigma'_v)$ for some $\sigma'_v$. Write

$$\sigma_v(\mathrm{Fr}_v) = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} \quad , \quad \sigma'_v(\mathrm{Fr}_v) = \begin{pmatrix} a'_v & 0 \\ 0 & b'_v \end{pmatrix}.$$

We wish to show these matrices are conjugate.

Let $w$ and $u$ be places of $E$ and $K$ above $v$ respectively. Since $BC_{E/F}(\pi) = \pi(\sigma_E)$ we have that $\sigma_v = \sigma'_v$ on $W_{E_w}$. In particular, since $\mathrm{Fr}_v^2 \in W_{E_w}$ we have $\sigma_v(\mathrm{Fr}_v^2) = \sigma'_v(\mathrm{Fr}_v^2)$ and,

$$\begin{pmatrix} a_v^2 & 0 \\ 0 & b_v^2 \end{pmatrix} \sim \begin{pmatrix} {a'_v}^2 & 0 \\ 0 & {b'_v}^2 \end{pmatrix}.$$

Similarly, $BC_{K/F}(\pi) = \pi(\sigma_K)$ implies that

$$\begin{pmatrix} a_v^3 & 0 \\ 0 & b_v^3 \end{pmatrix} \sim \begin{pmatrix} {a'_v}^3 & 0 \\ 0 & {b'_v}^3 \end{pmatrix},$$

Let us assume we actually have the equality

$$\begin{pmatrix} a_v^2 & 0 \\ 0 & b_v^2 \end{pmatrix} = \begin{pmatrix} {a'_v}^2 & 0 \\ 0 & {b'_v}^2 \end{pmatrix},$$

by interchanging $a'_v$ and $b'_v$ if necessary. The case where $a_v = a'_v$ is easy. For then the second and third powers of $b_v$ must equal those of $b'_v$, forcing $b_v = b'_v$. So in this case we are done. Similarly $b_v = b'_v$ implies $a_v = a'_v$, so suppose $a_v = -a'_v, b_v = -b'_v$. This forces the condition on the cubes of the matrices to be

$$\begin{pmatrix} a_v^3 & 0 \\ 0 & b_v^3 \end{pmatrix} = \begin{pmatrix} {b'_v}^3 & 0 \\ 0 & {a'_v}^3 \end{pmatrix},$$

i.e. $b_v = \zeta a'_v$ where $\zeta^3 = 1$. If $\zeta^3 \neq 1$, then

$$\sigma_v(\mathrm{Fr}_v) = \begin{pmatrix} -a'_v & 0 \\ 0 & \zeta a'_v \end{pmatrix} \quad \text{and} \quad (Ad \circ \sigma_v)(\mathrm{Fr}_v) = \begin{pmatrix} -\zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\zeta^2 \end{pmatrix}.$$

The latter matrix is an element of order 6 in $S_4 \subset \mathrm{PGL}_2(\mathbb{C})$, but $S_4$ has no elements of order 6. So we must have $\zeta = 1$, i.e. $a'_v = b_v$ and $b'_v = -b_v = -a'_v = a_v$, i.e. our original matrices are conjugate. □

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