DISTRIBUTION OF LOCAL SIGNS OF MODULAR FORMS AND MURMURATIONS OF FOURIER COEFFICIENTS

KIMBALL MARTIN

Abstract. Recently, we showed that global root numbers of modular forms are biased toward $+1$. Together with Pharis, we also showed an initial bias of Fourier coefficients towards the sign of the root number. First, we prove analogous results with respect to local root numbers.

Second, a subtle correlation between Fourier coefficients and global root numbers, termed murmurations, was recently discovered for elliptic curves and modular forms. We conjecture murmurations in a more general context of different (possibly empty) combinations of local root numbers.

Last, an appendix corrects a sign error in our joint paper with Pharis.

1. Introduction

Here we study the traces of Atkin–Lehner operators on spaces of newforms $S_k^{\mathrm{new}}(N) = S_k^{\mathrm{new}}(\Gamma_0(N))$. There are two main reasons we are interested in this: (1) to understand distributions of local root numbers of newforms; (2) to explore correlation of Fourier coefficients of newforms with respect to local root numbers, and in particular explore variations on recently discovered murmuration phenomena. These questions are local analogues of recent discoveries about global root numbers.

1.1. **Distributions of local root numbers.** In [Mar18a, Mar23], we observed a bias of newforms towards global root number $+1$, even though asymptotically these account for 50% of newforms. Namely, for fixed pair (k, N), outside of a prescribed set of exceptions, there are strictly more newforms in $S_k^{\mathrm{new}}(N)$ with root number $+1$ than -1. Moreover, the excess number of forms with root number $+1$ is essentially independent of k, and is typically an elementary factor times the class number of $\mathbb{Q}(\sqrt{-N})$.

In fact [Mar18a] was primarily concerned with the distribution of local root numbers in $S_k^{\mathrm{new}}(N)$. Suppose N is squarefree, and $q_1, \dots, q_m$ are primes dividing N. We obtained a criterion for when the local root numbers (i.e., Atkin–Lehner eigenvalues) at $q_1, \dots, q_m$ are perfectly equidistributed in $S_k^{\mathrm{new}}(N)$, i.e., the number of newforms with prescribed local signs at $q_1, \dots, q_m$ does not depend on the choice of signs. We also showed that there is a bias towards/away all local signs being -1, with the direction of the bias depending on the parities of $\frac{k}{2}$ and the number of prime divisors of N.

The motivation for studying the distribution of local root numbers in [Mar18a] was for applications to congruences mod 2. Suppose further that N is a squarefree product of an odd number of primes. In [Mar18b], we showed that, apart from levels of the form $N = 2p_1p_2$ when $k = 2$, if the local Atkin–Lehner signs are perfectly equidistributed for $q_1, \dots, q_m$, then for any newform $f \in S_k(N)$ and any prescribed choice of local signs at

Date: April 24, 2025.

$q_1, \dots, q_m$, there is a newform $g \in S_k(N)$ with those prescribed signs which is congruent to f mod 2. Further when $k = 2$, where there is a bias towards all local signs -1, one can do the same (without perfect equidistribution) when one prescribes the local signs of g to all be -1.

The approach in [Mar18a, Mar23] is via explicit trace formulas. However for general levels N and arbitrary collections of prime divisors $q_1, \dots, q_m$ of N, the trace formula is rather complicated, making a complete generalization of [Mar18a] difficult. Our first goal here is to study the distribution of the local root number at a single prime q for general levels N. It would be interesting to see if there are similar applications to mod 2 congruences as in [Mar18b] for general N, but we do not pursue this here. (However, see Proposition 7.1 for when quadratic twisting implies an analogous mod 2 congruence result.)

Let q be a prime, $r \geq 1$, and $M \geq 1$ such that $q \nmid M$. Denote by $S_k^{\mathrm{new}}(q^r M)^{\pm_q}$ the subspace of $S_k^{\mathrm{new}}(q^r M)$ which is the $\pm$-eigenspace of the Atkin–Lehner operator W_q at q. Define

$$\Delta_k(q^r, M) = \dim S_k^{\mathrm{new}}(q^r M)^{+_q} - \dim S_k^{\mathrm{new}}(q^r M)^{-_q}.$$

In other words, $\Delta_k(q^r, M)$ is the trace of W_q on $S_k^{\mathrm{new}}(q^r M)$. In Section 4 we obtain very explicit formulas for $\Delta_k(q^r, M)$ which are of the form

$$\Delta_k(q^r, M) = \begin{cases} C_1 h_{\mathbb{Q}(\sqrt{-q})} + \delta_{r=1} D_1 & \text{if } r \text{ is odd,} \\ C_2 + \delta_{r=2}(k-1) D_2 & \text{if } r \text{ is even.} \end{cases} \tag{1.1}$$

Here δ_* is the Kronecker delta, and C_i and D_i are elementary functions which depend on q and r, depend on the prime factorization of M (i.e., are expressible in terms of multiplicative functions of M), and only depend in a mild way on k. In general, these functions depend on $k \bmod 24$, and whether $k = 2$, but in most cases only involve a factor of $(-1)^{\frac{k}{2}}$. The explicit form of these functions breaks up into various cases, but for instance when $q \equiv 1 \bmod 4$, $r = 1$, and M is odd, we have

$$\Delta_k(q, M) = \frac{1}{2}(-1)^{\frac{k}{2}} \kappa_{-q}(M) h_{\mathbb{Q}(\sqrt{-q})} + \delta_{k=2}\mu(M),$$

where μ is the Möbius function, and κ_{-q} is the multiplicative function defined by (3.5).

As a consequence, outside of certain exceptional cases, we characterize when $\Delta_k(q^r, M) = 0$, and specify the sign of $\Delta_k(q^r, M)$ when it is nonzero. Let $\omega(M)$ be the number of primes dividing M, $\omega_1(M)$ the number of primes sharply dividing M, and $\omega_2(n; M)$ the number of $p^2 \parallel M$ such that $\left(\frac{n}{p}\right) = 1$.

Theorem 1.1 (odd exponent)**.** *Let q be a prime, $M \geq 1$ be coprime to q and write $M = 2^e M'$ where M' is odd. Let $r \geq 1$ be an odd integer. If $r \leq 3$ assume $q \geq 5$, and if $r = 1$ further assume that $k \geq 4$ or M is not squarefree.*

Then $\Delta_k(q^r, M) = 0$ if and only if (i) M' is not cubefree; (ii) $\left(\frac{-q}{p}\right) = 1$ for some $p \parallel M'$; (iii) $e \geq 5$; (iv) $e = 4$ and $q \equiv 1 \bmod 4$; or (v) $e = 1, 2, 3$ and $q \equiv 7 \bmod 8$.

Moreover, when $\Delta_k(q^r, M) \neq 0$, its sign is $(-1)^{k/2 + \omega_1(M') + \omega_2(-q, M')} b_{q,e}$, where $b_{q,e}$ is the sign of the quantity $\alpha_1(-q; e)$ in Table 2. I.e., we may take $b_{q,e} = +1$ if $e = 0$; -1 if $e = 1, 2$; $\left(\frac{-1}{q}\right)$ if $e = 3$; and $-\left(\frac{2}{q}\right)$ if $e = 4$. Hence this sign is simply $(-1)^{k/2 + \omega(M)}$ when M is squarefree.

Theorem 1.2 (even exponent ≥ 4)**.** *Let q be a prime, $r \geq 4$ even, and $M \geq 1$ coprime to q. Assume $q^r \neq 16$ and write $M = 2^e M'$ where M' is odd.*

Then $\Delta_k(q^r, M) = 0$ if and only if (i) M' is not cubefree, (ii) $16 \mid M$ or (iii) $p \equiv 1 \bmod 4$ for some $p \parallel M'$.

If $\Delta_k(q^r, M) \neq 0$, then its sign is $(-1)^{\frac{k}{2}+\omega_1(M')+\omega_2(-1;M')} b_{2,e}$, where $b_{2,e} = 1$ if $e = 0, 3$ and $b_{2,e} = -1$ if $e = 1, 2$. In particular, this sign is $(-1)^{\frac{k}{2}+\omega(M)}$ if M is squarefree.

See Section 4 for analogous results for other cases (e.g., when q^r is small, or when $r = 1$, $k = 2$ and M is squarefree). We remark that sometimes $\Delta_k(q^r, M) = 0$ is forced upon us by the action of quadratic twists—see Proposition 7.1—but quadratic twisting does not suffice to explain most cases of perfect equidistribution of local root numbers.

Due to the $\delta_{r=2}(k-1)D_2$ term in (1.1), the behavior is different when $r = 2$. In this case we describe the asymptotic behavior, which in the following setting asserts a bias towards local root number -1.

Proposition 1.3 (exponent 2)**.** *Fix a prime q, and consider $k + M \to \infty$ such that $k \geq 2$ is even and $M \geq 1$ is coprime to q. Then $\Delta_k(q^2, M) \to -\infty$. More precisely $\Delta_k(q^2, M) \sim \frac{1-k}{12}\kappa_\infty(M)$, where κ_∞ is the multiplicative function defined by (3.8).*

We also establish the asymptotic behavior in q under local conditions on M (see Proposition 4.4). The fact that local root number distributions behave differently in $r = 2$ parallels the fact that the bias of global root numbers is different for levels which are perfect squares (see [Mar23]). We do not have a compelling intuitive explanation for why this is (for local or global root numbers), but we do note that, when q is odd, $r = 2$ is precisely the case where the class of possible local representations π_q at q associated to newforms $f \in S_k^{\mathrm{new}}(q^r N)$ includes ramified principal series and ramified twists of Steinberg representations, all of which have local root number $\left(\frac{-1}{q}\right)$. However, such forms cannot account for the bias towards local root number -1 when $q \equiv 1 \bmod 4$.

We also remark that the fact that (1.1) only depends in a mild way on the weight implies the following.

Corollary 1.4 (boundedness in k)**.** *Fix a prime q and $r \geq 1$. Assume $r \neq 2$. Then $|\Delta_k(q^r, M)|$ is bounded as $k \to \infty$.*

This boundedness is also a simple consequence of existing trace formulas, but perhaps was not explicitly stated in the literature. In fact our formulas yield that $|\Delta_k(q^r, M)|$ is typically constant in k.

Remark 1.5. The corollary implies that, if $r \neq 2$, the trace of W_q on the q-new part of $S_k(q^r M)$ is bounded in k, and in fact only depends on k a mild way. One can view this as very strict equidistribution of the Atkin–Lehner sign at q in the weight aspect. A more refined problem is to study the distribution of q-adic Galois representations for modular forms at q; see recent work of Bergdall and Pollack [BP] taking $r = 1$.

1.2. **Correlation of initial Fourier coefficients with local signs.** In [MP22], we showed that for squarefree levels N, the trace of a Hecke operator T_ℓ on the subspace of forms in $S_k^{\mathrm{new}}(N)$ with root number $+1$ (resp., -1) is positive (resp., negative) for ℓ small relative to N.[1] In other words, for small ℓ, the sign of the Fourier coefficients $a_\ell(f)$ are

[1]There is a sign error for this result in [MP22] when $k \equiv 0 \bmod 4$. We correct this in Appendix A.

biased towards the sign of the root number of f. Combined with [Mar18a], this means that for small ℓ the Fourier coefficients $a_\ell(f)$ have a positive (resp., negative) bias for forms with the more common (resp., less common) global root number.

Here we obtain analogous results for local root numbers: for small ℓ, the Fourier coefficients $a_\ell(f)$ have a positive (resp., negative) bias for forms with the more common (resp., less common) local root number. However, in this case the bias only occurs under suitable congruence/divisibility conditions, and when these are not satisfied, there is essentially no bias for the a_ℓ's based on the local root number.

For simplicity, we restrict to levels of the form $N = qM$ where M is squarefree or twice a squarefree number.

Theorem 1.6 (bias of initial Fourier coefficients)**.** *Let q, ℓ denote primes such that $\ell < \frac{q}{4}$. Let M be a squarefree or twice a squarefree number which is coprime to $q\ell$. If $k = 2$, further assume $4 \mid M$. Suppose $\Delta_k(q, M) \neq 0$.*

(1) Either $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q$ is 0 or it has the same sign as $\Delta_k(q, M)$. It is 0 if and only if (i) $\left(\frac{-q\ell}{p}\right) = 1$ for some odd $p \mid M$ or (ii) M is even and $q\ell \equiv 7 \bmod 8$.

(2) If q is sufficiently large with respect to k, M, ℓ, and if $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q \neq 0$, then the trace of T_ℓ on $S_k^{\mathrm{new}}(qM)^{\pm_q}$ has the same sign as $\pm\Delta_k(q, M)$.

We remark that if the odd part of M is not squarefree, then the correlation between the signs of $\Delta_k(q, M)$ and $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q$ will alternate depending on quadratic residue symbols involving odd primes with $p^2 \parallel M$. On the other hand, if $p^3 \mid M$ for an odd p or $32 \mid M$, then one has perfect equidistribution of both local root numbers at q and trace of T_ℓ with respect to the local root number at q. See Section 5.1 for details.

Moreover, it is clear from our trace formulas that one can similarly treat levels of the form $N = q^r M$ with $r \geq 2$, and the behavior that occurs is similar to the case of $r = 1$.

1.3. **Murmurations.** Recently [HLOP] numerically discovered an oscillatory pattern, which they call murmurations, in averages of a_ℓ's (for ℓ prime) over elliptic curves of fixed rank or root number. Sutherland[2] did further extensive calculations—for elliptic curves, modular forms, and abelian surfaces—to help clarify and make precise the murmuration phenomena. The calculations for modular forms rely on the trace formula for $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} T_\ell W_N$. (One does not need to restrict to prime ℓ, but we will for simplicity.) We describe the phenomenon for modular forms, and then propose some generalizations, both for modular forms and elliptic curves.

Fix $k \geq 2$ even and let $\mathcal{F} = \mathcal{F}_k$ be a suitably large family of weight k newforms, say all weight k newforms (with trivial nebentypus), or all of those with squarefree level. Let $\mathcal{F}(X, Y)$ (resp. $\mathcal{F}^\pm(X, Y)$) be the set of newforms $f \in \mathcal{F}$ with level $X \leq N \leq Y$ (resp. and have have root number ± 1). Denote by $\mathcal{F}(X, Y)^{(\ell)}$ (resp. $\mathcal{F}^\pm(X, Y)^{(\ell)}$ be the subset of such forms with level N coprime to ℓ. The murmuration phenomena is the numerical observation that, for a fixed $\beta > 1$, the averages

$$A^\pm_{\mathcal{F}}(\ell, X; \beta) = \frac{1}{\#\mathcal{F}^\pm(X, \beta X)^{(\ell)}} \sum_{f \in \mathcal{F}^\pm(X, \beta X)^{(\ell)}} \ell^{1-\frac{k}{2}} a_\ell(f)$$

[2]See: https://math.mit.edu/~drew/murmurations.html

Figure 1. Weight 2 murmurations for W_N for squarefree levels $1000 \leq N \leq 2000$

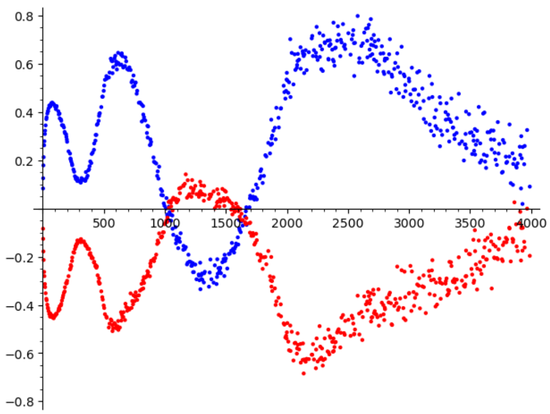
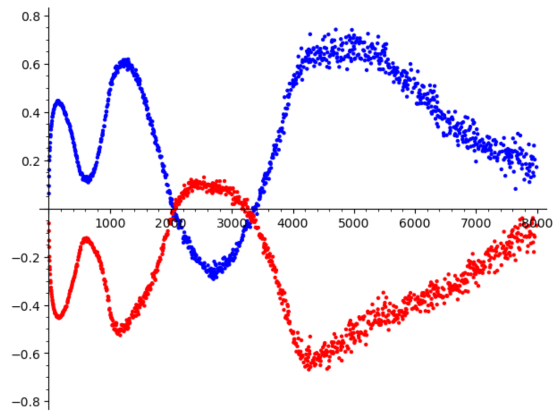


Figure 2. Weight 2 murmurations for W_N for squarefree levels $2000 \leq N \leq 4000$



tend to continuous *murmuration functions* $M^{\pm}_{\mathcal{F}}(x;\beta)$ as $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$. The restriction to forms in $\mathcal{F}^{\pm}(X,Y)^{(\ell)}$, as opposed to $\mathcal{F}^{\pm}(X,Y)$, is just a computational convenience and will not affect asymptotics (see Section 6.4). The normalization factor $\ell^{1-\frac{k}{2}}$ weights the Fourier coefficients so that each term is $O(\sqrt{\ell})$. In all numerical calculations, we follow Sutherland and take $\beta = 2$.

See Figs. 1 and 2 for plots of $A^{\pm}_{\mathcal{F}}(\ell, X; 2)$ for $X = 1000$ and $X = 2000$ for the weight 2 squarefree level family. Blue dots represent root number $+1$ and red dots -1. We labeled the horizontal axes by ℓ, but really one should think of the horizontal axes as being labeled by $\frac{\ell}{X}$, so both graphs have horizontal range $0 < \frac{\ell}{X} < 4$. That the graphs have a limit in this scale is called *scale invariance* in $\frac{\ell}{X}$. The limiting murmuration functions $M^{\pm}_{\mathcal{F}}$ oscillate infinitely [Zub].

To analyze these murmurations, since $M^{+}_{\mathcal{F}}(x;\beta) = -M^{-}_{\mathcal{F}}(x;\beta)$ (see Corollary 6.3) one can instead study a single weighted sum

$$A_{\mathcal{F}}(\ell, X; \beta) = \frac{1}{\#\mathcal{F}(X, \beta X)^{(\ell)}} \sum_{f \in \mathcal{F}(X,\beta X)^{(\ell)}} w(f) \ell^{1-\frac{k}{2}} a_\ell(f), \tag{1.2}$$

where $w(f)$ is the root number of f and $\mathcal{F}(X,Y) = \mathcal{F}^{+}(X,Y) \cup \mathcal{F}^{-}(X,Y)$. Then the assertion is that $A_{\mathcal{F}}(\ell, X; \beta) \to M_{\mathcal{F}}(x;\beta)$ as $\frac{\ell}{X} \to x$ where $M_{\mathcal{F}} = \frac{1}{2}(M^{+}_{\mathcal{F}} - M^{-}_{\mathcal{F}}) = M^{+}_{\mathcal{F}}$.

Zubrilina [Zub] proved such murmurations for $A_{\mathcal{F}}$ when $\mathcal{F}$ is the family of weight k newforms of squarefree level. In fact, Zubrilina proves a localized version of murmurations—essentially this means one can work with intervals of the form $[X, \beta X]$ where $\beta \to 0$ as $X \to \infty$. This leads to a continuous *murmuration density function*, and one obtains the murmuration functions $M_{\mathcal{F}}(x;\beta)$ by integrating the murmuration density function (and this implies continuity in β).

As trace formulas for more general Atkin–Lehner operators times Hecke operators, i.e., $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} T_\ell W_Q$ (where $Q = Q_N \mid N$), are in some ways quite similar to $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} T_\ell W_N$,

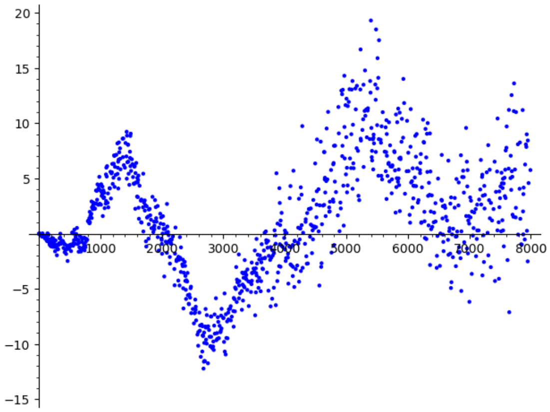


Figure 3. Weight 2 murmurations for $\sqrt{N}W_1$ for squarefree levels $2000 \le N \le 4000$

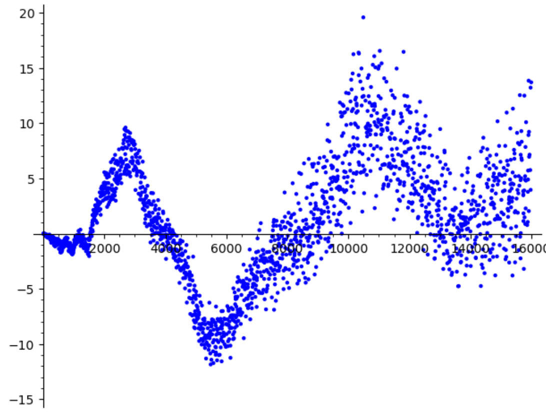


Figure 4. Weight 2 murmurations for $\sqrt{N}W_1$ for squarefree levels $4000 \le N \le 8000$

one might wonder if murmurations similarly exist with respect to Atkin–Lehner eigenvalues. Here one needs to choose how to vary Q along with N in these the averages.

One possibility is to consider averages of the form

$$A^{\mathcal{Q}}_{\mathcal{F}}(\ell, X; \beta) = \frac{1}{\#\mathcal{F}(X, \beta X)^{(\ell)}} \sideset{}{'}\sum_{X \le N \le \beta X} \sum_{f \in \mathcal{F}(N)} w_Q(f) \sqrt{\frac{N}{Q}} \ell^{1-\frac{k}{2}} a_\ell(f), \tag{1.3}$$

where $\mathcal{Q}$ is a sequence of divisors $Q \mid N$ for each level N appearing the family $\mathcal{F}$. Here $\mathcal{F}(N) = \mathcal{F}(N, N)$, and $w_Q(f)$ is the W_Q-eigenvalue of f. The notation $\sum'$ means we restrict to summing N coprime to ℓ. The normalization factor $\sqrt{\frac{N}{Q}}$ is included to keep the averages at about the same size so they do not tend to 0 as $X \to \infty$ (see Section 6.2).

Note that if each $Q = N$ then (1.3) becomes (1.2). At the other extreme if each $Q = 1$ then we are considering sums without any root numbers (as $w_1(f) = 1$). See Figs. 3 and 4 for plots of the averages in (1.3) in this case when $k = 2$. (Graphs for $k = 4$ are roughly similar.) Without the normalization factor of $\sqrt{\frac{N}{Q}} = \sqrt{N}$ in this case, the analogues of Figs. 3 and 4 are graphs which individually have similar shapes, but whose vertical scale shrinks with X. That the unweighted averages for $Q = 1$ tend to 0 reflects the root number symmetry $M^+_{\mathcal{F}}(x; \beta) = -M^-_{\mathcal{F}}(x; \beta)$.

Remark 1.7. One could also consider weighting the sums in (1.3) by signs, e.g., $(-1)^{\omega(N/Q)}$ in the squarefree case when $k \ge 4$, to account for the initial bias from Theorems 1.1 and 1.6. However in the situations we tested including this sign actually destroys the murmurations!

In Proposition 5.5, we write down a formula for $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} W_Q T_\ell$ (for simplicity for squarefree N) which is amenable to computing such averages. We used this to investigate murmurations with respect to Atkin–Lehner signs in a variety of settings. What seems important for the existence of such murmurations is that one considers a sequence of (N, Q)'s which are "arithmetically compatible". For instance, taking a sequence of

FIGURE 5. δ-smoothed version of Fig. 4 with $\delta = \frac{1}{2}$

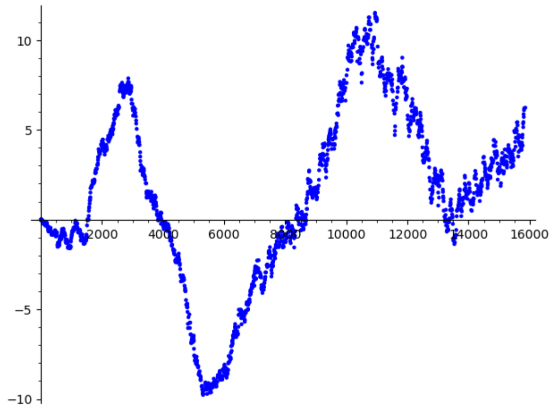
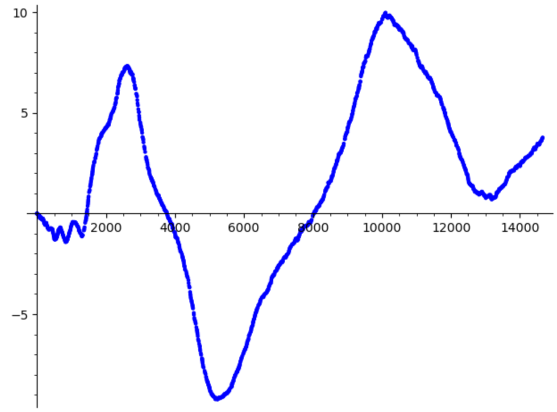


FIGURE 6. δ-smoothed version of Fig. 4 with $\delta = \frac{3}{4}$



(N, Q)'s where $Q \approx \sqrt{N}$, we numerically saw a random distribution of averages with no apparent murmurations.

Let $\mathbb{N}^{\mathrm{sqf}}$ denote the set of squarefree positive integers, and for $r \geq 1$ let $\mathbb{N}_r^{\mathrm{sqf}}$ be the subset of $\mathbb{N}^{\mathrm{sqf}}$ consisting of those with exactly r prime factors. We say a sequence of pairs $\{(N, Q)\}$ is *arithmetically compatible* in any of the following situations:

(I) $N = QM$ where M is constant and Q ranges over all elements of $\mathbb{N}^{\mathrm{sqf}}$ or $\mathbb{N}_r^{\mathrm{sqf}}$ such that $(M, Q) = 1$.

(II) $N = QM$ where Q is a squarefree constant and M ranges over all elements of $\mathbb{N}$, $\mathbb{N}^{\mathrm{sqf}}$ or $\mathbb{N}_r^{\mathrm{sqf}}$ such that $(M, Q) = 1$.

(III) Fix $r \geq 2$, let $0 \leq m < r$, and fix primes $p_1 < \cdots < p_m$. Let $N = p_1 \dots p_r$ range over elements of $\mathbb{N}_r^{\mathrm{sqf}}$ such that $p_1 < \cdots < p_r$, and $Q = p_{i_1} \dots p_{i_s}$ where $0 \leq s < r$ and $\{i_1, \dots, i_s\}$ is a fixed subset of $\{1, \dots, r\}$.

In all cases it is assumed the sequence $\{(N, Q)\}$ is arranged in order of increasing N.

For instance, when $r = 2$, Type III consists of sequences $\{(N, Q) = (p_1p_2, Q) : p_1 < p_2\}$, where we can choose to fix p_1 or not, and Q is taken to be one of the following 4 fixed forms: $1, p_1, p_2, p_1p_2$. Note that Type III includes the $\mathbb{N}_r^{\mathrm{sqf}}$ cases of Types I and II.

In Figs. 3 and 4 where $Q = 1$, and more generally for Type II and II graphs, it is not clear whether the averages $A^{\mathcal{Q}}$ should actually converge to a continuous function with fluctuations in very short intervals or whether there is some inherent "random noise." To be more confident the limiting graphs should exist, we consider the δ-smoothed averages

$$\tilde{A}_{\mathcal{F}}^{\mathcal{Q},\delta}(\ell, X; \beta) = \frac{1}{\#\{\ell' : \ell \leq \ell' < \ell + \ell^{\delta}\}} \sum_{\ell \leq \ell' < \ell + \ell^{\delta}} A_{\mathcal{F}}^{\mathcal{Q}}(\ell', Y; \beta).$$

See Figs. 5 and 6 for δ-smoothed versions of Fig. 4.

Conjecture 1.8 (Murmurations for Atkin–Lehner operators)**.** *Let $(\mathcal{N}, \mathcal{Q})$ be an arithmetically compatible sequence of (N, Q)'s of Type I, II or III as above, and fix a weight k. Let $\mathcal{F}$ the family of newforms which lie in $S_k^{\mathrm{new}}(N)$ for some $N \in \mathcal{N}$.*

Figure 7. Weight 4 murmurations for $N = 5Q$ squarefree

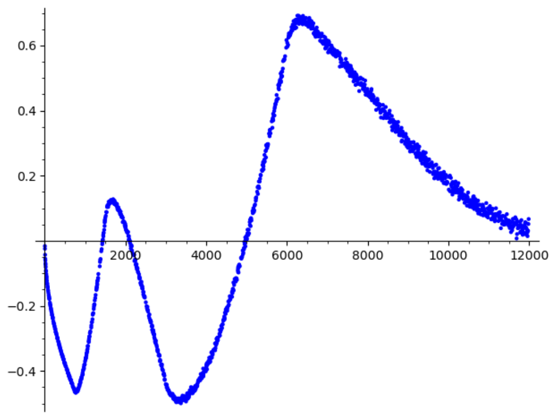
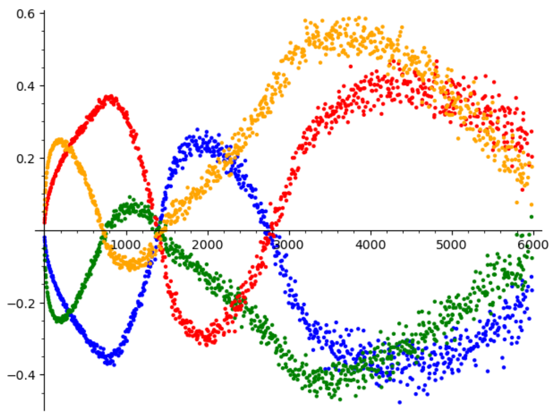


Figure 8. Murmurations for AL-eigenspaces on $S_2(2q)$



(1) *If $(\mathcal{N}, \mathcal{Q})$ is of Type I, then the averages $A_{\mathcal{F}}^{\mathcal{Q}}(\ell, X)$ have murmurations which are scale invariant in $\frac{\ell}{N}$. More precisely, as $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$,*

$$A_{\mathcal{F}}^{\mathcal{Q}}(\ell, X; \beta) \to M_{\mathcal{F}}^{\mathcal{Q}}(x; \beta)$$

for a murmuration function $M_{\mathcal{F}}^{\mathcal{Q}}$ which is continuous on $[0, \infty) \times (1, \infty)$.

(2) *If $(\mathcal{N}, \mathcal{Q})$ is of Type II or III, then for some $\delta < 1$ the δ-smoothed averages $\tilde{A}_{\mathcal{F}}^{\mathcal{Q},\delta}(\ell, X; \beta)$ have murmurations which are scale invariant in $\frac{\ell}{N}$. More precisely, as $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$,*

$$\tilde{A}_{\mathcal{F}}^{\mathcal{Q},\delta}(\ell, X; \beta) \to \tilde{M}_{\mathcal{F}}^{\mathcal{Q},\delta}(x; \beta)$$

for a murmuration function $\tilde{M}_{\mathcal{F}}^{\mathcal{Q},\delta}$ which is continuous on $[0, \infty) \times (1, \infty)$.

Recall that Type I murmurations with $N = Q$ squarefree and $k = 2$ are illustrated in Figs. 1 and 2. See Fig. 7 for a Type I plot with $N = 5Q$ (Q squarefree) and $k = 4$, averaging over the range $15000 < N < 30000$. Similarly, non-smoothed and smoothed Type II plots with $Q = 1$ are given in Figs. 3 to 6. Illustrations of Type III situations with $N = pq$ are presented in a different format in Figs. 8 to 10 by looking at Atkin–Lehner eigenspaces, as will be described below.

When $M = 1$ and $Q \in \mathbb{N}^{\mathrm{sqf}}$, Conjecture 1.8 follows from [Zub], and we expect that one can prove the general Type I case in a similar way. However, the types of sums that one needs to handle for Types II and III will require a different type of analysis. (For Type I and given x, there are only finitely many terms to consider from the trace formula, but for Types II and III there are an unbounded number of terms.) Here we merely show the following as evidence towards the above conjecture.

Theorem 1.9. *Assume $(\mathcal{N}, \mathcal{Q})$ is an arithmetically compatible family of pairs of Type I, where $\mathcal{N} = \{MQ : Q \in \mathbb{N}^{\mathrm{sqf}}, (Q, M) = 1\}$ for some fixed squarefree M. Then Conjecture 1.8(1) holds for $x < \frac{1}{4M} - \varepsilon$, for any $\varepsilon > 0$. Specifically $A_{\mathcal{F}}^{\mathcal{Q}}(\ell, X; \beta) \to c\sqrt{x} + \delta_{k=2} d$ in this range, for some constants $c = c_{\mathcal{F},\mathcal{Q},\beta}$ and $d = d_{\mathcal{F},\mathcal{Q},\beta}$.*

Figure 9. Murmurations for AL-eigenspaces on $S_4(2q)$

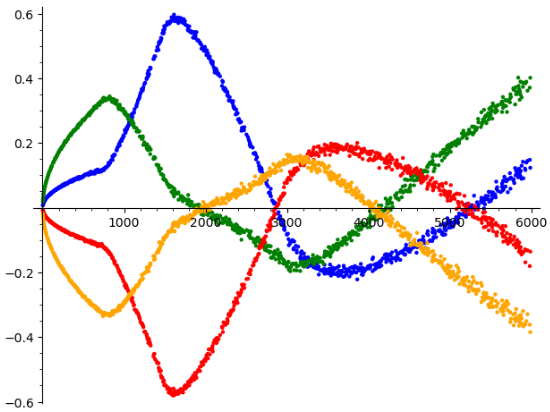
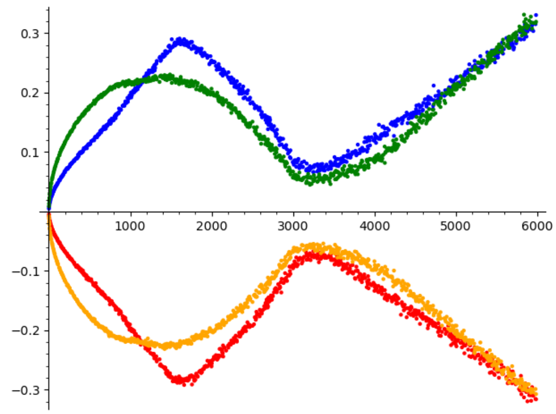


Figure 10. Murmurations for AL-eigenspaces on $S_4(pq)$



In particular, when $M = 1$, this says that the first $\frac{1}{16}$-th of the graphs in Figs. 3 and 4 are approximately of the form $c\sqrt{x} + d$. This agrees with [Zub], which also computes c, d. Similarly, the theorem asserts that the first $\frac{1}{25}$-th of the graph in Fig. 7 is approximately of the form $c\sqrt{x}$.

If one works with (squarefree or general) levels N that have a fixed number of prime divisors as in Type III, then one can alternatively look how the collection of Atkin–Lehner signs is correlated with Fourier coefficients. See Figs. 8 and 9 for a graph of averages of a_ℓ's over newforms in $S_2(pq)$ and $S_4(pq)$ with fixed Atkin–Lehner signs at p, q, where $p = 2$ and $3000 < q < 6000$. The blue and green dots correspond to signs `++` and `--` and red and orange dots to signs `+-` and `-+`, respectively. (The first sign denotes the sign at p, and the second the sign at q.) See Fig. 10 for the analogous graph for $S_4(pq)$ where p, q both vary such that $p < q$ and $6000 < pq < 12000$.

Conjecture 1.10 (Murmurations on Atkin–Lehner eigenspaces)**.** *Fix k, r and $0 \leq m < r$. Fix primes $p_1 < \cdots < p_m$. Let $\mathcal{N}$ be the set of levels $N = p_1 \dots p_r \in \mathbb{N}_r^{\mathrm{sqf}}$ such that $p_1 < p_2 < \cdots < p_r$. Let $\mathcal{F}$ be the family of weight k newforms with level in $\mathcal{N}$. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \pm 1^r$, let $\mathcal{F}^\varepsilon$ be the subset of $f \in \mathcal{F}$ with Atkin–Lehner sign ε_i at p_i, and consider the averages*

$$A^\varepsilon_{\mathcal{F}}(\ell, X; \beta) = \frac{1}{\#\mathcal{F}^\varepsilon(X, \beta X)} \sideset{}{'}\sum_{f \in \mathcal{F}^\varepsilon(X, \beta X)} \ell^{1-\frac{k}{2}} a_\ell(f). \tag{1.4}$$

These averages have murmurations which are scale invariant in $\frac{\ell}{N}$. That is, as $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$,

$$A^\varepsilon_{\mathcal{F}}(\ell, X; \beta) \to M^\varepsilon_{\mathcal{F}}(x; \beta)$$

for a murmuration function $M^\varepsilon_{\mathcal{F}}$ which is continuous on $[0, \infty) \times (1, \infty)$.

Remark 1.11. (1) As in the case of murmurations with respect to global root numbers, we expect these murmuration functions oscillate infinitely, and at least for

Type I that they arise from murmuration density functions which correspond to letting $\beta \to 0$ as $X \to \infty$.

(2) In the Type I case of Conjecture 1.8, there is no need to weight by $\sqrt{N/Q}$ as N/Q is constant. For Type II, one is weighting by a constant times $\sqrt{N}$, so by scale-invariance one could alternatively weight by $\sqrt{\ell}$. We chose to weight by $\sqrt{N/Q}$ as it seems to be the right order of normalization in general.
(3) When averaging Fourier coefficients over Atkin–Lehner eigenspaces $\mathcal{F}^\varepsilon$ in Conjecture 1.10, there is no need to weight by an analogue of $\sqrt{N/Q}$ (or consider smoothed averages), since the trace of T_ℓ on $\mathcal{F}^\varepsilon(N)$ is a linear combination of the traces of $T_\ell W_Q$ on $\mathcal{F}(N)$ where one sums over all $Q \mid N$. Correspondingly, we expect the murmuration functions to be different on each Atkin–Lehner eigenspace when $r = m + 1$, i.e., when all but one prime is fixed in the level as in Figs. 8 and 9.
(4) It is not clear whether the smoothed averages are actually needed in Conjecture 1.8(2), or how much smoothing is actually needed.
(5) One could also consider analogues where Q is not required to be squarefree.

See Section 6.3 for details on how Conjecture 1.10 is related to Conjecture 1.8. This relation implies that Theorem 1.9 also provides evidence for Conjecture 1.10.

Finally, one might wonder about analogues of Conjectures 1.8 and 1.10 in the original setting of elliptic curves. Earlier calculations of Sutherland indicate that there are no apparent murmurations if one does not weight by any root number; more generally our calculations also do not suggest any murmurations for elliptic curves in Type II situations.

However, numerically there appear to be murmurations in Type I situations, i.e., $N = QM$ with M fixed and Q varying, at least after smoothing. For instance, see Fig. 11 for a plot of $\tilde{A}^{\mathcal{Q}}_{\mathcal{E}}(\ell, X; 2)$ for the family with $N = 2Q$ squarefree, $X = 20000$ and $\beta = 2$, and Fig. 12 for the smoothed averages $\tilde{A}^{\mathcal{Q},\delta}_{\mathcal{E}}(\ell, X)$ with $\delta = 0.75$. For comparison, these averages (in blue) are plotted on top of the averages (1.2) weighted by global root numbers (in red). This suggests the following.

Conjecture 1.12 (Partial root number murmurations for elliptic curves)**.** *Let $(\mathcal{N}, \mathcal{Q})$ be an arithmetically compatible sequence of (N, Q)'s of Type I. Let $\mathcal{E}$ the set of rational newforms which lie in $S_2(N)$ for some $N \in \mathcal{N}$. Then for some $\delta < 1$, the smoothed averages $\tilde{A}^{\mathcal{Q},\delta}_{\mathcal{E}}(\ell, X; \beta)$ have murmurations which are scale invariant in $\frac{\ell}{N}$.*

As in the Type II and III cases for modular forms, it is not clear whether the smoothing in Conjecture 1.12 should be needed (even in the global root number case of $Q = N$).

1.4. **Additional remarks.** We checked the results stated in the introduction, as well as many of our formulas below, numerically in Sage [Sage] for a wide variety of small parameters.

Acknowledgements. I thank John Bergdall, Alex Cowan, Nina Zubrilina, and the anonymous referee for some helpful comments and discussions. Some of the computing for this project was performed at the OU Supercomputing Center for Education & Research (OSCER) at the University of Oklahoma (OU).

FIGURE 11. w_Q (blue) versus w_N (red) murmurations for elliptic curves of squarefree conductor $N = 2Q$, with $20000 < Q < 40000$

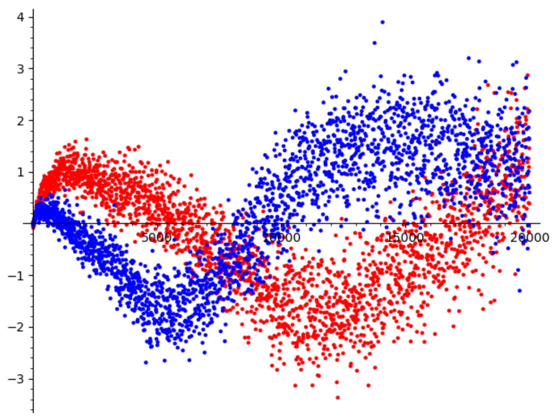
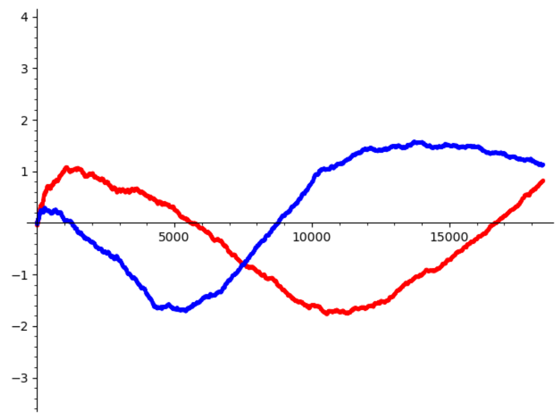


FIGURE 12. δ-smoothed analogues of Fig. 11 with $\delta = \frac{3}{4}$



2. NOTATION AND PRELIMINARIES

2.1. **Class numbers.** Let $\Delta = \lambda^2\Delta_0$, where Δ_0 is a negative fundamental discriminant. Let $h'(\Delta)$ be the weighted class number for primitive binary negative definite quadratic forms of discriminant Δ. Explicitly, $h'(-3) = \frac{1}{3}$, $h'(-4) = \frac{1}{2}$, and $h'(\Delta_0) = h(\Delta_0)$ is the usual class number if $\Delta_0 < -4$. Moreover,

$$h'(\lambda^2\Delta_0) = \gamma_{\Delta_0}(\lambda)h'(\Delta_0), \quad \gamma_{\Delta_0}(\lambda) = \sum_{d|\lambda} \mu(d)\left(\frac{\Delta_0}{d}\right)\frac{\lambda}{d}. \tag{2.1}$$

Since γ_{Δ_0} is a Dirichlet convolution of multiplicative functions, it is also multiplicative, and given on (nontrivial) prime powers by

$$\gamma_{\Delta_0}(p^m) = p^{m-1}\left(p - \left(\frac{\Delta_0}{p}\right)\right).$$

Let $\Delta \leq 0$ and $t \geq 1$. Let $H_1(\Delta) = H(\Delta)$ be the Hurwitz class number. This is defined by

$$H(\Delta) = \sum_{d|\lambda} h'(d^2\Delta_0),$$

where we write $\Delta = \lambda^2\Delta_0$ for a fundamental discriminant Δ_0. One deduces (e.g., [Mar23, (2.3)]) that

$$H(\lambda^2\Delta_0) = \eta_{\Delta_0}(\lambda)h'(\Delta_0), \quad \eta_{\Delta_0}(\lambda) = \sum_{d|\lambda} \mu(d)\left(\frac{\Delta_0}{d}\right)\sigma(\lambda/d). \tag{2.2}$$

Just like γ_{Δ_0}, we have that η_{Δ_0} is multiplicative and it is given on prime powers by

$$\eta_{\Delta_0}(p^m) = \sigma(p^m) - \left(\frac{\Delta_0}{p}\right)\sigma(p^{m-1}).$$

For $t \geq 2$, write $(t, \Delta) = a^2 b$ where b is squarefree, and put $\Delta' = \Delta/(t, \Delta)$, $t' = t/(t, \Delta)$. Set

$$H_t(\Delta) = \begin{cases} (t, \Delta)\left(\frac{\Delta'/b}{t'}\right) H(\Delta'/b) & \text{if } b \mid \Delta', \\ 0 & \text{else.} \end{cases}$$

2.2. **Other quantities arising in the trace formula.** Fix $s \in \mathbb{Z}$ and $\ell \geq 1$. Let $\rho, \bar{\rho}$ denote the roots of $x^2 - sX + \ell$. Define

$$p_k(s, \ell) = \begin{cases} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} & \text{if } s^2 \neq 4\ell, \\ (k-1)(\frac{s}{2})^{k-2} & \text{if } s^2 = 4\ell. \end{cases}$$

In particular, when $s = 0$, the roots $\rho, \bar{\rho}$ are $\pm\sqrt{-\ell}$, and

$$p_k(0, \ell) = (-\ell)^{\frac{k}{2}-1}.$$

We also remark that

$$p_2(s, \ell) = 1$$

and

$$p_k(s, \ell) = \ell^{\frac{k}{2}-1} U_{k-2}\left(\frac{s}{2\sqrt{\ell}}\right),$$

where $U_k(t)$ denotes the Chebyshev polynomial of the second kind.

Let $Q(n)$ be the greatest integer such that $Q(n)^2 \mid n$.

3. Traces on newspaces

Fix an even weight $k \geq 2$, a prime q and positive integers ℓ, M. Assume q, ℓ, M are pairwise coprime. The Atkin–Lehner operator W_q, defined as in [AL70], acts on $S_k(N)$. This action is taken to be the trivial action when $(q, N) = 1$.

For an integer $r \geq 0$, set

$$t(r; M) = \operatorname{tr}_{S_k(q^r M)} T_\ell W_q, \quad t^{\text{new}}(r; M) = \operatorname{tr}_{S_k^{\text{new}}(q^r M)} T_\ell W_q.$$

(Throughout the analysis in this section, q, ℓ, k will be fixed, so we suppress them from our notation for brevity.) For $r < 0$, we interpret these quantities to be 0. When $\ell = 1$, $t^{\text{new}}(r; M) = \Delta_k(q^r, M)$.

Fix a newform $g \in S_k(q^{r_0} M_0)$. Denote by $w_q(g)$ the eigenvalue for g under the action of W_q on $S_k(q^{r_0} M_0)$. Let $N = q^r M$, and assume that $r_0 \leq r$ and $M_0 \mid M$. Let π_g^N denote the subspace of $S_k(N)$ spanned by forms $g(dz)$ where $d \mid q^{r-r_0} M_0^{-1} M$. Then T_ℓ acts by a scalar on π_g^N. One computes the trace of W_q on π_g^N from [AL70, (5.1)–(5.2)], which yields

$$\operatorname{tr}_{\pi_g^N} T_\ell W_q = \begin{cases} \sigma_0(M/M_0) a_\ell(g) w_q(g) & \text{if } r \equiv r_0 \bmod 2, \\ 0 & \text{else.} \end{cases}$$

Thus

$$t(r; M) = \sum_{\substack{r_0 \leq r \\ r_0 \equiv r \bmod 2}} \sum_{M_0 \mid M} \sigma_0(M/M_0) t^{\text{new}}(r_0; M_0).$$

Hence

$$t(r; M) - t(r-2; M) = \sum_{M_0 \mid M} \sigma_0(M/M_0) t^{\text{new}}(r; M_0).$$

Since $\sigma_0 = 1 * 1$ (where $*$ denotes Dirichlet convolution) and the Dirichlet inverse of the constant function 1 is μ, the Dirichlet inverse of σ_0 is $\mu * \mu$, which is the multiplicative function defined by

$$(\mu * \mu)(p^m) = \begin{cases} -2 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m \geq 3. \end{cases}$$

Thus

$$t^{\text{new}}(r, M) = \sum_{d \mid M} (\mu * \mu)(d) \left(t(r; M/d) - t(r-2; M/d) \right).$$

Reorganizing the trace formula from [SZ88, (2.7)], we see that

$$t(r; M) = A_{1,0}(r; M) - \delta_{r \geq 2}\, A_{1,1}(r-2; M) + A_2(r; M) + A_3, \tag{3.1}$$

where

$$A_{1,\varepsilon}(r; M) = -\frac{1}{2} \sum_{\substack{s^2 \leq 4q^r \ell \\ q^{r+\varepsilon} \mid s}} p_k(q^{-r/2} s, \ell) \sum_{\substack{t \mid M \\ M/t \text{ squarefree}}} H_t(s^2 - 4q^r \ell),$$

$$A_2(r; M) = -\frac{1}{2} \delta_{r \text{ even}}\, \varphi(q^{r/2}) \sum_{\substack{\ell' \mid \ell \\ q^{r/2} \mid (\ell' + \ell/\ell')}} \min(\ell', \ell/\ell')^{k-1} \sum_{\substack{t \mid M \\ M/t \text{ squarefree}}} (Q(t), (\ell' - \ell/\ell')),$$

and

$$A_3 = A_3(r; M) = \delta_{k=2}\, \sigma(\ell).$$

Here and below the index s lies in $\mathbb{Z}$, whereas $t, \ell' \in \mathbb{Z}_{>0}$.

Hence

$$t(r; M) - t(r-2; M) = \begin{cases} A_{1,0}(r; M) + A_2(r; M) + A_3 & \text{if } r = 0, 1, \\ A_{1,0}(r; M) - A_{1,0}(r-2; M) - A_{1,1}(r-2; M) & \\ \quad + \delta_{r \geq 4} A_{1,1}(r-4; M) + A_2(r; M) - A_2(r-2; M) & \text{if } r \geq 2. \end{cases}$$

To compute $t^{\text{new}}(r; M)$, we want to compute the quantities

$$\tilde{A}_\star(r; M) := \sum_{d \mid M} (\mu * \mu)(d) A_\star(r; M/d) = ((\mu * \mu) * A_\star(r; \cdot))(M), \tag{3.2}$$

where $\star$ is any index.

Proposition 3.1. *With notation as above, we have*

$$t^{\text{new}}(r; M) = \begin{cases} \tilde{A}_{1,0}(r; M) + \tilde{A}_2(r; M) + \tilde{A}_3 & \text{if } r = 0, 1, \\ \tilde{A}_{1,0}(r; M) - \tilde{A}_{1,0}(r-2; M) - \tilde{A}_{1,1}(r-2; M) & \\ \quad + \delta_{r \geq 4} \tilde{A}_{1,1}(r-4; M) + \tilde{A}_2(r; M) - \tilde{A}_2(r-2; M) & \text{if } r \geq 2. \end{cases} \tag{3.3}$$

In the remainder of this section, we compute the quantities $\tilde{A}_\star(r; M)$ in the main situations of interest for us.

3.1. $A_{1,\varepsilon}$ **sums.** Let $\varepsilon \in \{0, 1\}$.

	Δ_0	λ	$H(-4q^r\ell)$	$H(-q^r\ell)$
r even, $\ell \not\equiv 3 \bmod 4$	-4ℓ	$q^{r/2}$	$\eta_{-4\ell}(q^{r/2})h'(-4\ell)$	0
r even, $\ell \equiv 3 \bmod 4$	$-\ell$	$2q^{r/2}$	$\eta_{-\ell}(2q^{r/2})h'(-\ell)$	$\eta_{-\ell}(q^{r/2})h'(-\ell)$
r odd, $q\ell \not\equiv 3 \bmod 4$	$-4q\ell$	$q^{(r-1)/2}$	$\eta_{-4q\ell}(q^{(r-1)/2})h'(-4q\ell)$	0
r odd, $q\ell \equiv 3 \bmod 4$	$-q\ell$	$2q^{(r-1)/2}$	$\eta_{-q\ell}(2q^{(r-1)/2})h'(-q\ell)$	$\eta_{-q\ell}(q^{(r-1)/2})h'(-q\ell)$

Table 1. Hurwitz class numbers by case

3.1.1. *Computing $\tilde{A}_{1,\varepsilon}(r;M)$ when $4\ell < q^{r+2\varepsilon}$.* Suppose $4\ell < q^{r+2\varepsilon}$. Then only the $s=0$ term occurs in the outer sum for $A_{1,\varepsilon}(r;M)$, and we have

$$A_{1,\varepsilon}(r;M) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} \sum_{\substack{t \mid M \\ M/t \text{ squarefree}}} H_t(-4q^r\ell).$$

We assume $(M, q\ell) = 1$, so for $t \mid M$ we have $(t, -4q^r\ell) = 1$ unless t is even, in which case $(t, -4q^r\ell) = 2$ or 4. Hence

$$H_t(-4q^r\ell) = \begin{cases} \left(\frac{-q^r\ell}{t}\right) H(-4q^r\ell) & \text{if } t \text{ odd,} \\ 2\left(\frac{-q^r\ell}{t/2}\right) H(-q^r\ell) & \text{if } t \equiv 2 \bmod 4, \\ 4\left(\frac{-q^r\ell}{t/4}\right) H(-q^r\ell) & \text{if } t \equiv 0 \bmod 4. \end{cases} \tag{3.4}$$

In the following analysis, we will assume ℓ is squarefree. Write $-4q^r\ell = \lambda^2\Delta_0$, where Δ_0 is a fundamental discriminant. We can rewrite $H(-4q^r\ell)$ and $H(-q^r\ell)$ in terms of $h'(\Delta_0)$ on a case-by-case basis as in Table 1. We will use this to calculate $\tilde{A}_1$ in cases.

The following computation will be useful. First, for an integer Δ, note that $\left(\frac{\Delta}{\cdot}\right) * |\mu|$ is the multiplicative function given on nontrivial prime powers by

$$\left(\left(\frac{\Delta}{\cdot}\right) * |\mu|\right)(p^m) = \begin{cases} 1 + \left(\frac{\Delta}{p}\right) & \text{if } p \nmid \Delta \text{ or } m = 1, \\ 0 & \text{if } p \mid \Delta \text{ and } m = 1. \end{cases}$$

Consequently $\kappa_\Delta := (\mu * \mu) * (\left(\frac{\Delta}{\cdot}\right) * |\mu|)$ is the multiplicative function given on nontrivial prime powers by

$$\kappa_\Delta(p^m) = \begin{cases} \left(\frac{\Delta}{p}\right) - 1 & \text{if } m = 1, \\ -\left(\frac{\Delta}{p}\right) & \text{if } p \nmid \Delta \text{ and } m = 2, \\ -1 & \text{if } p \mid \Delta \text{ and } m = 2, \\ 1 & \text{if } p \mid \Delta \text{ and } m = 3, \\ 0 & \text{else.} \end{cases} \tag{3.5}$$

When M is odd, this is simple: $H_t(-4q^r\ell)$ will be given by the first case of (3.4). From this we see

$$A_{1,\varepsilon}(r;\cdot) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} H(-4q^r\ell) \cdot \left(\frac{-q^r\ell}{\cdot}\right) * |\mu| \qquad (M \text{ odd})$$

Therefore

$$\tilde{A}_{1,\varepsilon}(r;M) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} H(-4q^r\ell)\, \kappa_{-q^r\ell} \qquad (M \text{ odd}).$$

For general M, we will evaluate $\tilde{A}_{1,\varepsilon}$ by separating it into 3 sums as follows:

$$\tilde{A}_{1,\varepsilon}(r;M) = \sum_{\substack{d|M \\ d \text{ odd}}} (\mu*\mu)(d) A_{1,\varepsilon}(r;M/d) + \sum_{\substack{d|M \\ d\equiv 2 \bmod 4}} (\mu*\mu)(d) A_{1,\varepsilon}(r;M/d) + \sum_{\substack{d|M \\ d\equiv 4 \bmod 8}} (\mu*\mu)(d) A_{1,\varepsilon}(r;M/d)$$

We will also separate the sum in $A_{1,\varepsilon}$ over t according to $v_2(t)$. Write $M = 2^e M'$ where M' is odd. Then

$$A_{1,\varepsilon}(r;M) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} \sum_{\substack{t|M' \\ M'/t \text{ squarefree}}} \left(H_{2^e t}(-4q^r\ell) + H_{2^{e-1}t}(-4q^r\ell)\right),$$

where we interpret H_t to be 0 if $t \notin \mathbb{Z}$. Hence

$$\tilde{A}_{1,\varepsilon}(r;M) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} \sum_{j=0}^{2} \sum_{d|M'} (\mu*\mu)(2^j d) \sum_{\substack{t|(M'/d) \\ M'/dt \text{ squarefree}}} \left(H_{2^{e-j}t}(-4q^r\ell) + H_{2^{e-j-1}t}(-4q^r\ell)\right).$$

Rewriting (3.4) with index $2^e t$ for t odd, we have

$$H_{2^e t}(-4q^r\ell) = \begin{cases} \left(\frac{-q^r\ell}{t}\right) H(-4q^r\ell) & \text{if } e=0, \\ 2\left(\frac{-q^r\ell}{t}\right) H(-q^r\ell) & \text{if } e=1, \\ 4\left(\frac{-q^r\ell}{t}\right)\left(\frac{-q^r\ell}{2}\right)^{e-2} H(-q^r\ell) & \text{if } e \geq 2. \end{cases}$$

Applying this to the previous formula yields

$$\tilde{A}_{1,\varepsilon}(r;M) = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} \alpha_1(-q^r\ell;e) \kappa_{-q^r\ell}(M'), \quad M = 2^e M' \text{ with } M' \text{ odd}, \tag{3.6}$$

where

$$\alpha_1(-q^r\ell;e) = \begin{cases} H(-4q^r\ell) & \text{if } e=0, \\ 2H(-q^r\ell) - H(-4q^r\ell) & \text{if } e=1,2, \\ \left(4\left(\frac{-q^r\ell}{2}\right) - 6\right) H(-q^r\ell) + H(-4q^r\ell) & \text{if } e=3, \\ \left(2 - 4\left(\frac{-q^r\ell}{2}\right)\right) H(-q^r\ell) & \text{if } e=4, \\ 0 & \text{if } e \geq 5. \end{cases}$$

3.1.2. *Computing $\tilde{A}_{1,0}(r;M)$ when $\ell = 1$.* Assume $\ell = 1$. Then Section 3.1.1 computes $\tilde{A}_{1,0}(r;M)$ in all cases except when $q^r \leq 4$, where there are terms with $s \neq 0$ in $A_{1,0}(r;\cdot)$. Namely, when $q^r = 1$ there are terms for $s = 0, \pm 1, \pm 2$, and when $q^r \in \{2,3,4\}$ there are terms for $s = 0, \pm q^r$.

Note that

$$p_k(\pm 1, 1) = \frac{\zeta_6^{k-1} + \zeta_3^{k-1}}{\zeta_6 + \zeta_3} = \begin{cases} -1 & \text{if } k \equiv 0 \bmod 6, \\ 1 & \text{if } k \equiv 2 \bmod 6, \\ 0 & \text{if } k \equiv 4 \bmod 6; \end{cases}$$

$$p_k(\pm 2, 1) = k-1;$$

$$p_k(\pm\sqrt{2},1)=\begin{cases}-1 & \text{if } k\equiv 0,6 \bmod 8,\\ 1 & \text{if } k\equiv 2,4 \bmod 8;\end{cases}$$

and

$$p_k(\pm\sqrt{3},1)=\begin{cases}-1 & \text{if } k\equiv 0,8 \bmod 12,\\ 1 & \text{if } k\equiv 2,6 \bmod 12,\\ 2 & \text{if } k\equiv 4 \bmod 12,\\ -2 & \text{if } k\equiv 10 \bmod 12.\end{cases}$$

<u>Case 1: $q^r=1$</u>

Suppose $r=0$. Note that

$$\begin{aligned}A_{1,0}(0;M)&=\frac{1}{2}\sum_{\substack{t\mid M\\ M/t \text{ squarefree}}}\left((-1)^{\frac{k}{2}}H_t(-4)-2p_k(1,1)H_t(-3)-2(k-1)H_t(0)\right)\\ &=\frac{1}{12}\sum_{\substack{t\mid M\\ M/t \text{ squarefree}}}\left(3(-1)^{\frac{k}{2}}\left(\frac{-4}{t}\right)-4p_k(1,1)\left(\frac{-3}{t}\right)+(k-1)t\right).\end{aligned}$$

Hence

$$\tilde{A}_{1,0}(0;M)=\frac{1}{12}\left(3(-1)^{\frac{k}{2}}\kappa_{-4}(M)-4p_k(1,1)\kappa_{-3}(M)+(k-1)\kappa_\infty(M)\right), \tag{3.7}$$

where $\kappa_\infty=(\mu*\mu)*(\mathrm{id}*|\mu|)$, which is the multiplicative function given by

$$\kappa_\infty(p^m)=\begin{cases}p-1 & \text{if } m=1,\\ p^2-p-1 & \text{if } m=2,\\ p^{m-3}(p-1)^2(p+1) & \text{if } m\geq 3.\end{cases} \tag{3.8}$$

<u>Case 2: $q^r=2$</u>

Suppose $q=2$ and $r=1$. Then

$$A_{1,0}(1;M)=\sum_{\substack{t\mid M\\ M/t \text{ squarefree}}}\left(\frac{(-1)^{\frac{k}{2}}}{2}H_t(-8)-p_k(\sqrt{2},1)H_t(-4)\right).$$

By assumption $t\mid M$ implies t is odd, so

$$\tilde{A}_{1,0}(1;M)=\frac{1}{2}\left((-1)^{\frac{k}{2}}\kappa_{-2}(M)-p_k(\sqrt{2},1)\kappa_{-1}(M)\right). \tag{3.9}$$

<u>Case 3: $q^r=3$</u>

Suppose $q=3$ and $r=1$. Then

$$A_{1,0}(1;M)=\sum_{\substack{t\mid M\\ M/t \text{ squarefree}}}\left(\frac{(-1)^{\frac{k}{2}}}{2}H_t(-12)-p_k(\sqrt{3},1)H_t(-3)\right).$$

Hence by the same argument as above, we have

$$\tilde{A}_{1,0}(1;M)=\left(\frac{(-1)^{\frac{k}{2}}}{2}\alpha_1(-3;e)\kappa_{-3}(M')-\frac{1}{3}p_k(\sqrt{3},1)\kappa_{-3}(M)\right), \tag{3.10}$$

where $M = 2^e M'$ with M' odd. We note that $\alpha_1(-3;e) = \frac{4}{3}, -\frac{2}{3}, -2, 2, 0$ for $e = 0$, $e = 1, 2$, $e = 3$, $e = 4$ and $e \geq 5$, respectively.

Case 4: $q^r = 4$

Suppose $q = 2$ and $r = 2$. Then

$$A_{1,0}(2;M) = \sum_{\substack{t|M \\ M/t \text{ squarefree}}} \left(\frac{(-1)^{\frac{k}{2}}}{2} H_t(-16) - (k-1)H_t(0) \right)$$
$$= \frac{1}{12} \sum_{\substack{t|M \\ M/t \text{ squarefree}}} \left(9(-1)^{\frac{k}{2}} \left(\frac{-1}{t} \right) + (k-1)t \right),$$

whence

$$\tilde{A}_{1,0}(2;M) = \frac{1}{12} \left(9(-1)^{\frac{k}{2}} \kappa_{-1}(M) + (k-1)\kappa_\infty(M) \right). \tag{3.11}$$

3.1.3. *Computing $\tilde{A}_{1,1}(r;M)$ when $\ell = 1$.* Assume $\ell = 1$. Then Section 3.1.1 computes $\tilde{A}_{1,1}(r;M)$ except in the case that $q = 2$ and $r = 0$, so suppose this. One sees

$$A_{1,1}(0;M) = \sum_{\substack{t|M \\ M/t \text{ squarefree}}} \left(\frac{(-1)^{\frac{k}{2}}}{2} H_t(-4) - (k-1)H_t(0) \right)$$
$$= \frac{1}{12} \sum_{\substack{t|M \\ M/t \text{ squarefree}}} \left(3(-1)^{\frac{k}{2}} \left(\frac{-1}{t} \right) + (k-1)t \right).$$

This implies

$$\tilde{A}_{1,1}(0;M) = \frac{1}{12} \left(3(-1)^{\frac{k}{2}} \kappa_{-1}(M) + (k-1)\kappa_\infty(M) \right). \tag{3.12}$$

3.2. **A_2 sums when $\ell = 1$.** Suppose $\ell = 1$. Then

$$A_2(r;M) = -\frac{1}{2} \delta_{r \text{ even}} \delta_{q^r|4} \sum_{\substack{t|M \\ M/t \text{ squarefree}}} Q(t).$$

The sum on the right is in fact $(Q * |\mu|)(M)$, which is a multiplicative function of M. Consequently,

$$\alpha_2 = (\mu * \mu) * (Q * |\mu|)$$

is a multiplicative function, and one can check that it is given on (nontrivial) prime powers by

$$\alpha_2(p^m) = \begin{cases} 0 & \text{if } m \text{ is odd}, \\ p-2 & \text{if } m = 2, \\ p^{\frac{m-4}{2}}(p-1)^2 & \text{if } m \geq 4 \text{ is even}. \end{cases}$$

Hence, when $\ell = 1$,

$$\tilde{A}_2(r;M) = \begin{cases} -\frac{1}{2}\alpha_2(M) & \text{if } q^r \in \{1,4\} \text{ and } M \in \square, \\ 0 & \text{else}. \end{cases} \tag{3.13}$$

e	$q^r \equiv 1 \bmod 4$	$q^r \equiv 3 \bmod 4$	$q = 2$
0	$H(-4q^r)$	$(3 - (\frac{-q}{2}))H(-q^r)$	$H(-2^{r+2})$
$1,2$	$-H(-4q^r)$	$\left((\frac{-q}{2}) - 1\right) H(-q^r)$	—
3	$H(-4q^r)$	$3\left((\frac{-q}{2}) - 1\right) H(-q^r)$	—
4	0	$2\left(1 - 2(\frac{-q}{2})\right) H(-q^r)$	—
≥ 5	0	0	—

Table 2. Computing $\alpha_1(-q^r; e)$ by cases

3.3. A_3 **sums.** Since $A_3(r; M) = \delta_{k=2}\,\sigma(\ell)$ is independent of r and M,

$$(\mu * \mu) * A_3(r; \cdot) = \delta_{k=2}\,\sigma(\ell)(\mu * \mu * 1) = \delta_{k=2}\,\sigma(\ell) \cdot \mu.$$

In other words,

$$\tilde{A}_3 = \tilde{A}_3(r; M) = \delta_{k=2}\,\mu(M)\sigma(\ell). \tag{3.14}$$

4. Dimension formulas

Let q, r, M, k be as in the previous section. Here we put together our calculations of $\tilde{A}_\star(r; M)$ when $\ell = 1$ to compute

$$\Delta_k(q^r, M) = t^{\mathrm{new}}(r; M) = \dim S_k^{\mathrm{new}}(q^r M)^{+_q} - \dim S_k^{\mathrm{new}}(q^r M)^{-_q}$$

Since one knows a formula for $\dim S_k^{\mathrm{new}}(q^r M) = \dim S_k^{\mathrm{new}}(q^r M)^{+_q} + \dim S_k^{\mathrm{new}}(q^r M)^{-_q}$ ([Mar05]), this will imply a formula for

$$\dim S_k^{\mathrm{new}}(q^r M)^{\pm_q} = \pm\frac{1}{2}\left(\dim S_k^{\mathrm{new}}(q^r M) \pm \Delta_k(q^r, M)\right).$$

We will use the following explicit calculations of κ_Δ and α_1.

Since $(-q^r, M) = 1$, we have

$$\kappa_{-q^r}(M) = \begin{cases} \prod_{p^2 \| M}\left(-(\frac{-q^r}{p})\right)\prod_{p \| M}\left((\frac{-q^r}{p}) - 1\right) & \text{if } M \text{ is cubefree,} \\ 0 & \text{if } p^3 \mid M \text{ for some } p. \end{cases}$$

In particular $\kappa_{-q^r}(M) = 0$ if and only if (i) $(\frac{-q^r}{p}) = 1$ for some $p \| M$; or (ii) $v_p(M) \geq 3$ for some p. Assuming neither (i) nor (ii) hold, then $\kappa_{-q^r}(M) = (-1)^{\omega_2(-q^r; M)}(-2)^{\omega_1(M)}$, where $\omega_1(M)$ is the number of primes sharply dividing M and $\omega_2(n; M)$ is the number of $p^2 \| M$ such that $(\frac{n}{p}) = 1$.

We also tabulate the values of $\alpha_1(-q^r; e)$ by cases in Table 2. These calculations use the fact that $H(-4q^r) = (3 - (\frac{-q}{2}))H(-q^r)$ when $-q^r \equiv 1 \bmod 4$. In particular, one sees that for q odd, $\alpha_1(-q^r; e) = 0$ if and only if (i) $q^r \equiv 1 \bmod 4$ and $e \geq 4$; (ii) $q \equiv 3 \bmod 8$, r is odd and $e \geq 5$; or (iii) $q \equiv 7 \bmod 8$, r is odd and $e > 0$. We note that $\alpha_1(-q^r; e) \leq 0$ if (a) $e \neq 1, 2$ or (b) $e = 3$, r is odd, and $q \equiv 7 \bmod 8$. Otherwise $\alpha_1(-q^r; e) \geq 0$.

4.1. **Dimensions for $r = 1$.** First suppose $r = 1$. Then $\Delta_k(q, M) = \tilde{A}_{1,0}(r; M) + \tilde{A}_3$ by (3.3).

When $q \geq 5$,

$$\Delta_k(q, M) = \frac{1}{2}(-1)^{\frac{k}{2}}\alpha_1(-q; e)\kappa_{-q}(M') + \delta_{k=2}\mu(M), \quad M = 2^e M' \text{ with } M' \text{ odd.}$$

Otherwise

$$\Delta_k(q, M) = \begin{cases} \frac{1}{2}\left((-1)^{\frac{k}{2}}\kappa_{-2}(M) - p_k(\sqrt{2}, 1)\kappa_{-1}(M)\right) + \delta_{k=2}\mu(M) & \text{if } q = 2, \\ \left(\frac{(-1)^{\frac{k}{2}}}{2}\alpha_1(-3; e)\kappa_{-3}(M') - \frac{1}{3}p_k(\sqrt{3}, 1)\kappa_{-3}(M)\right) + \delta_{k=2}\mu(M) & \text{if } q = 3. \end{cases}$$

Using these formulas, we can derive a precise elementary characterization of when the Atkin–Lehner signs at q are perfectly equidistributed.

Proposition 4.1. *Let q be a prime, $M \geq 1$ be coprime to q. Write $M = 2^e M'$, where M' is odd.*

(1) Suppose $q \geq 5$ and either $k \geq 4$ or M is not squarefree. Then $\Delta_k(q, M) = 0$ if and only if (i) M' is not cubefree; (ii) $\left(\frac{-q}{p}\right) = 1$ for some $p \parallel M'$; (iii) $16 \mid M$ and $q \equiv 1 \bmod 4$; (iv) $32 \mid M$ and $q \equiv 3 \bmod 8$; or (v) $v_2(M) \neq 0, 4$ and $q \equiv 7 \bmod 8$.

(2) Suppose $q \geq 5$, $k = 2$ and M is squarefree. Then $\Delta_k(q, M) = 0$ if and only if (i) $M = 1$ and $q \in \{5, 7, 13, 17\}$; or (ii) $M = 2$ and $q \in \{5, 11, 13, 19, 37, 43, 67, 163\}$.

(3) Suppose $q = 2$. We have $\Delta_k(2, M) = 0$ if and only if (i) M is not cubefree; (ii) $\prod_{p^2 \parallel M}\left(\frac{2}{p}\right)$ is -1 if $k \equiv 0, 2 \bmod 8$ and $+1$ if $k \equiv 4, 6 \bmod 8$; or (iii) $k = 2$ and $M = 1$ or $M \equiv 3, 5 \bmod 8$ is prime.

(4) Suppose $q = 3$. If $k \geq 4$ or M is not squarefree, then $\Delta_k(3, M) = 0$ if and only if (i) M' is not cubefree; (ii) $\left(\frac{-3}{p}\right) = 1$ for some $p \parallel M'$; (iii) $32 \mid M$; (iv) M is odd and $k \equiv 4, 10 \bmod 12$; or (v) $4 \parallel M$ and $k \not\equiv 4, 10 \bmod 12$. If $k = 2$ and M is squarefree, then $\dim S_k^{\mathrm{new}}(3M)^{+_3} = \dim S_k^{\mathrm{new}}(3M)^{-_3}$ if and only if $M = 1, 2$.

Proof. Case (1) follows immediately from the above vanishing conditions for $\alpha_1(-q; e)$.

Case (2): Now suppose $q \geq 5$, $k = 2$ and M is squarefree. Then $\Delta_k(q, M) = 0$ if and only if $\frac{1}{2}\alpha_1(-q; e)\kappa_{-q}(M') = \mu(M)$.

For a discriminant $\Delta < -4$, $H(\Delta) \geq 1$. Also if $\Delta = \lambda^2\Delta_0$ where Δ_0 is a fundamental discriminant and $\lambda > 1$, then $H(\Delta) \geq 2H(\Delta_0)$. Thus the only integers Δ such that $H(\Delta) = 1$ are $\Delta = \{-7, -8, -11, -19, -43, -67, -163\}$. Similarly, there are 19 discriminants $\Delta < 0$ with $H(\Delta) = 2$ (the minimal one is $\Delta = -427$).

If $\kappa_{-q}(M') \neq 0$, it equals $(-2)^{\omega(M')}$. Thus we can only have $\Delta_k(q, M) = 0$ if (i) $M' = 1$ and $\alpha_1(-q; e) = \pm 2$, or (ii) $M' = p$ is prime and $\alpha_1(-q; e) = \pm 1$.

When $M = 1$, $\Delta_k(q, M) = 0$ if and only if $H(-4q) = 2$, i.e., if $q \in \{5, 7, 13, 17\}$. When $M = 2$, $\Delta_k(q, M) = 0$ if and only if $H(-4q) = 2H(-q) + 2$, i.e., if and only if $q \in \{5, 11, 13, 19, 37, 43, 67, 163\}$. When $M = p \geq 3$, $\Delta_k(q, M) = 0$ if and only if $H(-4q) = 1$ and $\left(\frac{-q}{p}\right) = -1$, which never happens. When $M = 2p$, for a prime $p \geq 3$, $\Delta_k(q, M) = 0$ if and only if $2H(-q) = H(-4q) + 1$, which also never happens. This finishes case (2).

Case (3): Next suppose $q = 2$, and $k \neq 2$ or M is not squarefree. Then $t^{\mathrm{new}}(1; M) = 0$ if and only if $\kappa_{-2}(M) = \kappa_{-1}(M) = 0$ or $(-1)^{k/2}p_k(\sqrt{2}, 1) = \prod_{p^2 \parallel M}\left(\frac{2}{p}\right)$. The former never happens. The latter condition means $\prod_{p^2\|M}\left(\frac{2}{p}\right)$ is -1 if $k\equiv 0,2 \bmod 8$ and $+1$ if $k\equiv 4,6 \bmod 8$.

If $q=2$, $k=2$ and M is squarefree, then one needs $\kappa_{-2}(M)+\kappa_{-1}(M)=2\mu(M)$, which is true when $M=1$ or M is a prime $p\equiv 3,5 \bmod 8$. This proves case (3).

Case (4): Finally suppose $q=3$. Then $\Delta_k(q,M)=c\kappa_{-3}(M')+\delta_{k=2}\mu(M)$, where $c=\frac{1}{2}(-1)^{k/2}\alpha_1(-3;e)-\frac{1}{3}p_k(\sqrt{3},1)\kappa_{-3}(2^e)$. One checks that $c\in\{-1,0,1\}$, and $c=0$ if and only if (i) $e\geq 5$; (ii) $e=0$ and $k\equiv 4,10 \bmod 12$; or (iii) $e=2$ and $k\equiv 0,2,6,8 \bmod 12$.

Hence if $k\neq 2$ or M is not squarefree, then $\Delta_k(q,M)=0$ if and only if one of (i)–(iii) above holds; (iv) M' is not cubefree; or (v) $\left(\frac{-3}{p}\right)=1$ for some odd $p\parallel M$.

Now suppose $k=2$ and M is squarefree. Then $c=(-1)^{e+1}$, and we see $\mu(M)=c\kappa_{-3}(M')$ if and only if $M=1,2$. This completes case (4). $\square$

When the Atkin–Lehner signs are not perfectly equidistributed, it is also easy to give conditions for which Atkin–Lehner eigenspace is larger, and give bounds on the differences of dimensions. For simplicity, we only explicitly do the former when $q\geq 5$.

Proposition 4.2. *Let $q\geq 5$ be a prime, $M\geq 1$ be coprime to q, and $e=v_2(M)$. Put $\tilde{e}=0$ if (i) $e=0$, (ii) $e=3$ and $q\equiv 1 \bmod 4$, or (iii) $e\geq 4$ and $q\equiv 3 \bmod 8$. Let $\tilde{e}=1$ otherwise. Then*

$$\begin{cases}\Delta_k(q,M)\geq 0 & \text{if } \frac{k}{2}+\omega_1(M)+\omega_2(-q;M)+\tilde{e}\equiv 0 \bmod 2,\\ \Delta_k(q,M)\leq 0 & \text{else.}\end{cases}$$

Proof. The sign of $\Delta_k(q,M)$ agrees with the sign of $(-1)^{\frac{k}{2}}\alpha_1(-q;e)\kappa_{-q}(M')$. This is immediate from the above expression for $\Delta_k(q,M)$ unless $k=2$ and M is squarefree. In that situation it follows as $|\alpha_1(-q;e)\kappa_{-q}(M')|\geq|\mu(M)|=1$. $\square$

In particular, the above two propositions contain the $r=1$ case of Theorem 1.1.

4.2. **Dimensions for $r\geq 3$ odd.** Suppose $r\geq 3$ is odd. Then

$$\Delta_k(q^r,M)=\tilde{A}_{1,0}(r;M)-\tilde{A}_{1,0}(r-2;M)-\tilde{A}_{1,1}(r-2;M)+\delta_{r\geq 5}\tilde{A}_{1,1}(r-4;M).$$

First assume $q^{r-2}>4$, i.e., $q>3$ or $r\geq 5$. Then

$$\begin{aligned}\Delta_k(q^r,M)&=\frac{1}{2}(-1)^{\frac{k}{2}}\left(\alpha_1(-q^r;e)\kappa_{-q^r}(M')-2\alpha_1(-q^{r-2};e)\kappa_{-q^{r-2}}(M')+\delta_{r\geq 5}\alpha_1(-q^{r-4};e)\kappa_{-q^{r-4}}(M')\right)\\&=\frac{1}{2}(-1)^{\frac{k}{2}}\kappa_{-q}(M')\left(\alpha_1(-q^r;e)-2\alpha_1(-q^{r-2};e)+\delta_{r\geq 5}\alpha_1(-q^{r-4};e)\right).\end{aligned}$$

Recall $\kappa_{-q}(M')=0$ if and only if M' is not cubefree or $\left(\frac{-q}{p}\right)=1$ for some $p\parallel M'$.

Consider the factor $\aleph=\alpha_1(-q^r;e)-2\alpha_1(-q^{r-2};e)+\delta_{r\geq 5}\alpha_1(-q^{r-4};e)$. According to Table 2, we can write

$$\aleph=c_1\left(H(\Delta_0 q^{r-1})-2H(\Delta_0 q^{r-3})+\delta_{r\geq 5}H(\Delta_0 q^{r-5})\right),$$

where $\Delta_0=-4q$ if $q\equiv 1,2 \bmod 4$ and $\Delta_0=-q$ if $q\equiv 3 \bmod 4$. Here $c_1\in\{0,\pm 1,\pm 2,4,-6\}$ according to the cases in Table 2 (and $\left(\frac{-q}{2}\right)$ when $q\equiv 3 \bmod 4$). In particular, $c_1=0$ if and only if (a) $e\geq 5$, (b) $e=4$ and $q\equiv 1 \bmod 4$, or (c) $e=1,2,3$ and $q\equiv 7 \bmod 8$.

Moreover, by Table 1 and (2.2), we see that

$$\begin{aligned}\aleph &= c_1\left(\eta_{\Delta_0}(q^{\frac{r-1}{2}}) - \eta_{\Delta_0}(2q^{\frac{r-3}{2}}) + \delta_{r\geq 5}\eta_{\Delta_0}(q^{\frac{r-5}{2}})\right)h'(\Delta_0)\\ &= c_1\left(\sigma(q^{\frac{r-1}{2}}) - \sigma(2q^{\frac{r-3}{2}}) + \delta_{r\geq 5}\sigma(q^{\frac{r-5}{2}})\right)h'(\Delta_0).\end{aligned}$$

In particular, one deduces that $\aleph = 0$ if and only if $c_1 = 0$. This proves the following.

Proposition 4.3. *Let q be a prime, $r \geq 3$ odd, $M \geq 1$ be coprime to q, and write $M = 2^e M'$, where M' is odd. Assume $q^r \neq 8, 27$. Then $\Delta_k(q^r, M) = 0$ if and only if (i) M' is not cubefree; (ii) $\left(\frac{-q}{p}\right) = 1$ for some $p \parallel M'$; (iii) $e \geq 5$; (iv) $e = 4$ and $q \equiv 1 \bmod 4$; or (v) $e = 1, 2, 3$ and $q \equiv 7 \bmod 8$.*

Further, when $\Delta_k(q^r, M) \neq 0$, its sign is the sign of $(-1)^{k/2}\alpha_1(-q^r; e)\kappa_{-q^r}(M')$.

This completes the proof of Theorem 1.1.

Explicit equidistribution criteria for $q^r = 8$ and $q^r = 27$ break up into more cases based on the weight and, in the case of $q^r = 8$, a more delicate relation among quadratic residue symbols. We merely write down $t^{\mathrm{new}}(r; M)$ in these cases:

When $q^r = 8$, we have

$$\Delta_k(8, M) = \frac{1}{2}\left((-1)^{\frac{k}{2}}\kappa_{-2}(M) + p_k(\sqrt{2}, 1)\kappa_{-1}(M)\right).$$

When $q^r = 27$, we have

$$\Delta_k(27, M) = \left(\frac{1}{2}(-1)^{\frac{k}{2}}(\alpha_1(-27; e) - 2\alpha_1(-3; e)) + \frac{1}{3}p_k(\sqrt{3}, 1)\kappa_{-3}(2^e)\right)\kappa_{-3}(M').$$

4.3. **Dimensions for $r = 2$.** Suppose $r = 2$. Then

$$\Delta_k(q^2, M) = \tilde{A}_{1,0}(2; M) - \tilde{A}_{1,0}(0; M) - \tilde{A}_{1,1}(0; M) + \tilde{A}_2(2; M) - \tilde{A}_2(0; M).$$

There does not appear to be a clean characterization of when the Atkin–Lehner sign at q is perfectly equidistributed for general q, M, k, but we can give asymptotics for $t^{\mathrm{new}}(2; M)$ in various parameters. Let

$$b_{2,e} = \begin{cases} 1 & \text{if } e = 0, 3,\\ -1 & \text{if } e = 1, 2,\\ 0 & \text{if } e \geq 4.\end{cases}$$

Proposition 4.4. *(1) Fix q. Let k, M denote varying integers such that $k \geq 2$ is even and $M \geq 1$ is coprime to q. As $k + M \to \infty$, we have*

$$\Delta_k(q^2, M) \sim \frac{1}{12}(1-k)\kappa_\infty(M),$$

so in particular $\Delta_k(q^2, M) \to -\infty$.

(2) Fix k, M. Suppose M or $\frac{M}{2}$ is a cubefree integer such that $p \not\equiv 1 \bmod 4$ for each $p \parallel M$. Then as $q \to \infty$ along a sequence of primes not dividing M, we have

$$\Delta_k(q^2, M) \sim \frac{1}{4}(-1)^{k/2} b_{2,e}\kappa_{-1}(M')q.$$

In particular, for large q, the sign of $\Delta_k(q^2, M)$ is $(-1)^{\frac{k}{2} + b_{2,e} + \omega_1(M') + \omega_2(-1; M')}$.

The first part of this proposition coincides with Proposition 1.3.

Proof. Recall that $\tilde{A}_2(r;M)=0$ unless M is a square and $q^r\in\{1,4\}$, in which case it is $-\frac{1}{2}\alpha_2(M)$. Note that $0\le\alpha_2(M)<\sqrt{M}$. This combined with our analysis below will imply that the $\tilde{A}_2$ terms in $\Delta_k(q^2,M)$ will not contribute to the main asymptotics in either case.

Case (1): Note that for an integer Δ, we have $|\kappa_\Delta(M)|\le 2^{\omega_1(M)}$. Thus the sum $\tilde{A}_{1,0}(0;M)+\tilde{A}_{1,1}(0;M)$ equals $\frac{2^{\delta_{q=2}}}{12}(k-1)\kappa_\infty(M)$ plus terms that (in absolute value) are $O(2^{\omega_1(M)})$. The other terms in $\Delta_k(q^2,M)$ are also $O(2^{\omega_1(M)})$, except that when $q^r=4$ there is also a $\frac{1}{12}\kappa_\infty(M)$ term that cancels out half of the $\kappa_\infty(M)$ contribution from $\tilde{A}_{1,0}(0;M)+\tilde{A}_{1,1}(0;M)$. Since $\kappa_\infty(M)\ge\prod_{p|M}(p-1)$, the asymptotic in (1) follows.

Case (2): When $q\neq 2$, we have

$$\tilde{A}_{1,0}(2;M)-\tilde{A}_{1,1}(0;M)=\frac{1}{4}(-1)^{k/2}b_{2,e}(q+1-\left(\frac{-1}{q}\right))\kappa_{-1}(M').$$

The hypothesis guarantees that this is nonzero and grows like the asserted multiple of q, whereas all other terms in $\Delta_k(q^2,M)$ are bounded independent of q. $\square$

4.4. **Dimensions for $r\ge 4$ even.** Now suppose $r\ge 4$ is even. Then

$$\Delta_k(q^r,M)=\tilde{A}_{1,0}(r;M)-\tilde{A}_{1,0}(r-2;M)-\tilde{A}_{1,1}(r-2;M)+\tilde{A}_{1,1}(r-4;M)+\frac{\delta_{q^r=16}}{2}\alpha_2(M).$$

When $q^r=16$, we get

$$\Delta_k(16,M)=\frac{1}{2}\left((-1)^{\frac{k}{2}}\kappa_{-1}(M)+\alpha_2(M)\right),$$

and the $\alpha_2(M)$ term dominates asymptotically if $M\to\infty$ along a sequence of squares. If M is not a square, then $\alpha_2(M)=0$ so $\Delta_k(16,M)=0$ if and only if M is not cubefree or if $p\equiv 1 \bmod 4$ for some $p\parallel M$.

Now assume $q^r\neq 16$. Then

$$\Delta_k(q^r,M)=\frac{1}{2}(-1)^{\frac{k}{2}}\kappa_{-1}(M')\,\aleph,$$

where

$$\aleph=\alpha_1(-q^r;e)-2\alpha_1(-q^{r-2};e)+\alpha_1(-q^{r-4};e).$$

From Tables 1 and 2, we compute

$$\aleph=\begin{cases}2^{\frac{r-4}{2}} & \text{if } q=2,\\ \frac{1}{2}q^{\frac{r-4}{2}}(q-1)(q-\left(\frac{-1}{q}\right)) & \text{if } q\neq 2 \text{ and } e=0,3,\\ -\frac{1}{2}q^{\frac{r-4}{2}}(q-1)(q-\left(\frac{-1}{q}\right)) & \text{if } q\neq 2 \text{ and } e=1,2,\\ 0 & \text{if } e\ge 4.\end{cases}$$

This gives an explicit formula for $t^{\mathrm{new}}(r;M)$, which proves Theorem 1.2.

5. Correlation of Fourier coefficients and local signs

Now we will investigate the correlation of Fourier coefficients with Atkin–Lehner signs. For simplicity, we will only work with Atkin–Lehner operators at primes q that sharply divide the level. On the other hand, we will consider not just Atkin–Lehner operators W_q at a single prime q, but $W_Q=\prod_{q|Q}W_q$ for some squarefree $Q\ge 1$.

Let Q, ℓ, M be pairwise coprime positive integers with Q squarefree. Let $k \geq 2$ be even. From [SZ88], we have

$$\operatorname{tr}_{S_k(QM)} T_\ell W_Q = A_{1,0}(1; M) + \delta_{Q=1} A_2(0; M) + A_3$$

where $A_\star$ is defined as in Section 3 with Q in place of q. In particular, replacing s with $\frac{s}{Q}$ in the definition of $A_{1,\varepsilon}(r; M)$ we have

$$A_{1,0}(1; M) = -\frac{1}{2} \sum_{s^2 \leq \frac{4\ell}{Q}} p_k(s\sqrt{Q}, \ell) \sum_{\substack{t \mid M \\ M/t \text{ squarefree}}} H_t(s^2Q^2 - 4Q\ell).$$

We also have

$$\operatorname{tr}_{S_k^{\mathrm{new}}(QM)} T_\ell W_Q = \tilde{A}_{1,0}(1; M) + \delta_{Q=1} \tilde{A}_2(0; M) + \tilde{A}_3, \tag{5.1}$$

where as before $\tilde{A}_*(r; M) = \sum_{d \mid M} (\mu * \mu)(d) A_*(r; M/d)$. Here $\tilde{A}_3$ is given by (3.14).

5.1. **Traces for small ℓ.** The traces of $T_\ell W_Q$ are simpler when ℓ is small relative to Q. In particular, suppose that $4\ell < Q$ so that only the $s = 0$ term contributes to $A_{1,0}(1; M)$. The analysis in Section 3.1.1 also applies if we replace q by Q, and one has that

$$\operatorname{tr}_{S_k^{\mathrm{new}}(QM)} T_\ell W_Q = -\frac{1}{2}(-\ell)^{\frac{k}{2}-1} \alpha_1(-Q\ell; e) \kappa_{-Q\ell}(M') + \delta_{k=2} \mu(M) \sigma(\ell). \tag{5.2}$$

We have the following consequences for $Q = q$ and ℓ both prime. As before, write $M = 2^e M'$ with M' odd.

Proposition 5.1. *Assume $\ell < \frac{q}{4}$ is prime. Suppose either M' is not cubefree or $32 \mid M$. Then the trace of T_ℓ on $S_k^{\mathrm{new}}(qM)^{\pm_q}$ is independent of the sign $\pm_q$ for $\ell < \frac{q}{4}$. Equivalently, since $\Delta_k(q, M) = 0$, the average of $a_\ell(f)$ over newforms in $S_k^{\mathrm{new}}(qM)^{+_q}$ is equal to that for $S_k^{\mathrm{new}}(qM)^{-_q}$ for primes $\ell < \frac{q}{4}$.*

For simplicity, now assume M is cubefree. (Note that $\operatorname{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q = \delta_{k=2} \mu(M) \sigma(\ell)$ if $\ell < \frac{q}{4}$ and M' is not cubefree.) Then

$$\kappa_{-q\ell}(M') = \prod_{p \| M'} \left(\left(\frac{-q\ell}{p} \right) - 1 \right) \prod_{p^2 \| M'} \left(\frac{-q\ell}{p} \right)$$

and

$$\alpha_1(-q\ell; e) = \begin{cases} H(-4q\ell) & \text{if } e = 0, \\ -H(-4q\ell) & \text{if } e = 1, 2 \text{ and } q\ell \equiv 1 \bmod 4, \\ \left(\left(\frac{-q\ell}{2} \right) - 1 \right) H(-4q\ell) & \text{if } e = 1, 2 \text{ and } q\ell \equiv 3 \bmod 4, \end{cases}$$

which implies the following.

Proposition 5.2. *Suppose M is cubefree and $\ell < \frac{q}{4}$ is prime. If either (i) $\left(\frac{-q\ell}{p}\right) = 1$ for some $p \parallel M'$ or (ii) $e = 1, 2$ and $q\ell \equiv 7 \bmod 8$, then*

$$\operatorname{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q = \delta_{k=2} \mu(M)(\ell + 1).$$

Otherwise

$$\operatorname{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q = c_{1,e}(q\ell)(-2)^{\omega_1(M')-1}(-\ell)^{\frac{k}{2}-1} \prod_{p^2 \| M'} \left(\frac{-q\ell}{p} \right) H(-4q\ell) + \delta_{k=2} \mu(M)(\ell+1),$$

where

$$c_{1,e}(q\ell) = \begin{cases} 1 & \text{if } e = 0, \\ -1 & \text{if } e = 1,2 \text{ and } q\ell \equiv 1 \bmod 4, \\ -2 & \text{if } e = 1,2 \text{ and } q\ell \equiv 1 \bmod 8. \end{cases}$$

Now we compare the sign of this $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q$ (or whether it is 0 or not) with the sign of $\Delta_k(q,M)$. For simplicity, assume $k \geq 4$ or M is not squarefree so that the $\delta_{k=2}\mu(M)\sigma(\ell)$ term vanishes. Then from Proposition 4.1, we have $\Delta_k(q,M) = 0$ if and only if (i) $\left(\frac{-q}{p}\right) = 1$ for some $p \parallel M'$ or (ii) M is even and $q \equiv 7 \bmod 8$.

Corollary 5.3. *Suppose M is cubefree, $\ell < \frac{q}{4}$ is prime, and either $k \geq 4$ or M is not squarefree. If the quantities $\Delta_k(q,M) \neq 0$ and $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q \neq 0$ are both nonzero, then their ratio has sign $\prod_{p^2 \parallel M'} \left(\frac{\ell}{p}\right)$. In particular, their signs are the same if M' is squarefree.*

This corollary implies the first part of Theorem 1.6.

From [MS10, Proposition 14], we have

$$\left|\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell\right| \leq \ell^{(k-1)/2}\left(8\ell^3 4^{\omega(qM)} + \ell^{3/2}\right).$$

In particular, for fixed ℓ, k, M, we see that $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell$ is bounded independent of q. Since

$$\mathrm{tr}_{S_k^{\mathrm{new}}(qM)^{\pm q}} T_\ell = \frac{1}{2}\left(\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q \pm \mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell\right),$$

for q large $H(-4q\ell)$ dominates $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell$. Thus the sign of $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)^{\pm q}} T_\ell$ will be ± 1 times the sign of $\mathrm{tr}_{S_k^{\mathrm{new}}(qM)} T_\ell W_q$ for large q such that the latter trace is nonzero.

This proves the second part of Theorem 1.6.

Remark 5.4. One could also prove analogous statements for $T_\ell W_Q$. The restriction to $Q = q$ was simply because our goal here was to compare $T_\ell W_q$ with $\Delta_k(q,M)$.

5.2. **Traces for general ℓ with M squarefree.** Now, assuming M is squarefree, we give a formula for $\mathrm{tr}_{S_k^{\mathrm{new}}(QM)} T_\ell W_Q$ which is amenable to computation.

Since we restrict to squarefree levels, we specify certain multiplicative functions $\xi_\Delta^\star$ below only on squarefree integers. To apply standard Dirichlet convolution, we may view these as multiplicative functions on $\mathbb{N}$ which are, for instance, 0 on non-squarefree numbers.

For fixed Δ and squarefree t, it is straightforward to check that

$$H_t(\Delta) = \xi_\Delta^0(t) H(\Delta),$$

where ξ_Δ^0 is a multiplicative function satisfying

$$\xi_\Delta^0(p) = \begin{cases} \left(\frac{\Delta}{p}\right) & \text{if } p^2 \nmid \Delta, \\ p\frac{H(\Delta/p^2)}{H(\Delta)} & \text{if } p^2 \mid \Delta. \end{cases}$$

Then for squarefree M,

$$\sum_{t \mid M} H_t(\Delta) = (1 * \xi_\Delta^0)(M) \cdot H(\Delta) = \xi_\Delta^1(M) H(\Delta),$$

where ξ_Δ^1 is a multiplicative function such that $\xi_\Delta^1(p) = 1 + \xi_\Delta^0(p)$.

Write

$$\tilde{A}_{1,0}(1;M) = -\frac{1}{2}\sum_{s^2\le\frac{4\ell}{q}} p_k(s\sqrt{q},\ell)B_1(M,q(s^2q-4\ell)),$$

where

$$B_1(M;\Delta) = \sum_{d|M}(-2)^{\omega(d)}\sum_{t|M/d} H_t(\Delta).$$

The above shows that

$$B_1(M,\Delta) = ((\mu*\mu)*\xi^1_\Delta)(M)H(\Delta) = \xi_\Delta(M)H(\Delta),$$

where ξ_Δ is a multiplicative function satisfying $\xi_\Delta(p) = \xi^0_\Delta(p) - 1$. Using (2.2), we can explicitly write

$$\xi_\Delta(p) = \begin{cases} \left(\frac{\Delta}{p}\right) - 1 & \text{if } p^2 \nmid \Delta, \\ \frac{(p-1)(\left(\frac{\Delta_0}{p}\right)-1)}{(p^{e+1}-1)-\left(\frac{\Delta_0}{p}\right)(p^e-1)} & \text{if } p^2 \mid \Delta,\ 2e = v_p(\Delta/\Delta_0), \end{cases} \tag{5.3}$$

where Δ_0 is the negative fundamental discriminant dividing Δ.

When $Q=1$ and ℓ prime, we have

$$A_2(0;N) = -\sum_{t|N}(Q(t),\ell-1) = -\sigma_0(N).$$

This yields

$$\tilde{A}_2(0;N) = -\delta_{N=1}.$$

In summary, we have the following.

Proposition 5.5. *Let Q, M, ℓ be pairwise coprime integers such $N = QM$ is squarefree. If $Q = 1$, further assume that ℓ is prime. Then*

$$\operatorname{tr}_{S_k^{\mathrm{new}}(QM)} T_\ell W_Q = -\frac{\ell^{\frac{k}{2}-1}}{2}\sum_{s^2\le\frac{4\ell}{Q}} U_{k-2}\Big(\frac{s}{2}\sqrt{\frac{Q}{\ell}}\Big)\xi_{s^2Q^2-4Q\ell}(M)H(s^2Q^2-4Q\ell)$$
$$-\delta_{N=1} + \delta_{k=2}\mu(M)\sigma(\ell).$$

Remark 5.6. When $Q = 1$, this is the squarefree case of the trace formula for T_ℓ in [MS10]. When $M = 1$, this is the trace formula used in [Zub]. Assaf [Ass] also gives a trace formula for $T_\ell W_Q$ on the newspace (without a squarefree level assumption) which involves multiple summations. The point of Proposition 5.5 is to give the trace as an explicit linear combination of a minimal collection of class numbers.

6. Murmurations

6.1. **Analysis for Type I.** Let us now investigate murmurations for arithmetically compatible sequences $(\mathcal{N},\mathcal{Q}) = \{(N,Q)\}$ of Type I. For simplicity, we will assume $N = QM$ is squarefree where M is fixed and Q ranges over squarefree numbers coprime to M. Fix k and let $\mathcal{F}$ be the family of weight k newforms of a squarefree level $N \in \mathcal{N}$. Then

$$A^{\mathcal{Q}}_{\mathcal{F}}(\ell,X;\beta) = \frac{\sqrt{M}}{\#\mathcal{F}(X,\beta X)}\ell^{1-\frac{k}{2}}\sideset{}{'}\sum_{\frac{X}{M}\le Q\le\beta\frac{X}{M}} \operatorname{tr}_{S_k^{\mathrm{new}}(QM)} T_\ell W_Q, \tag{6.1}$$

where the prime on the sum means Q is restricted to squarefree numbers coprime to ℓM. Here $\beta > 1$ is fixed.

We want to consider the limit of these averages as $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$ for some $x \in [0, \infty)$ by substituting the trace formula from Proposition 5.5 into (6.1). Specifically, Conjecture 1.8 asserts that the limit exists. Note that one only gets s-terms appearing for $s^2 \leq \frac{4\ell}{Q} \leq \frac{4\ell M}{X} \sim 4Mx$. Consequently, we will only see a bounded number of s-terms. As $\#\mathcal{F}(X, \beta X)$ grows like a multiple of X^2 (see [Zub, Section 3.4] for a precise estimate), the $\delta_{N=1}$ term will contribute nothing asymptotically, and it is easy to see that the $\delta_{k=2}$ term will asymptotically contribute a constant as $\frac{\ell}{X} \to x$.

Hence it suffices to consider the finitely many s-terms from Proposition 5.5. The contribution from each of these terms is determined in [Zub] in when $M = 1$. Here we content ourselves with the more modest goal of analyzing the contribution from the $s = 0$ term, with the expectation that the work in [Zub] can be similarly modified to prove Conjecture 1.8 in this setting (as well as the variant where Q is restricted to $\mathbb{N}_r^{\mathrm{sqf}}$).

The $s = 0$ contribution to (6.1) is

$$\frac{(-1)^{\frac{k}{2}}}{2} \cdot \frac{\sqrt{M}}{\#\mathcal{F}(X, \beta X)} \sideset{}{'}\sum_{\frac{X}{M} \leq Q \leq \beta \frac{X}{M}} \xi_{-4Q\ell}(M) H(-4Q\ell). \tag{6.2}$$

Since $\xi_{-4Q\ell}(M) = \xi_{-4Q\ell}(2^{v_2(M)}) \prod_{\text{odd } p|M} \left(\left(\frac{-Q\ell}{p}\right) - 1\right)$, this value only depends on $Q\ell$ mod $8M$.

Lemma 6.1. *Fix integers $a, m \geq 1$ such that (a, m) is squarefree. Then there exists $c > 0$ such that*

$$\sideset{}{'}\sum_{\substack{1 \leq Q < X \\ Q\ell \equiv a \bmod m}} H(-4Q\ell) = cX\sqrt{\ell X} + O(X^{\frac{13}{10}+\varepsilon} + \sqrt{\ell} \log \ell),$$

uniformly in ℓ, X.

Proof. We may assume $Q\ell > 3$. Then $H(-4Q\ell) = h(-4Q\ell) + h(-Q\ell)$. Thus the lemma amounts to estimating class number sums in congruence classes with a squarefree restriction on Q. This is similar to classical class number averages, but now with a congruence condition. Lavrik [Lav71] has already determined class number moments over arithmetic progressions. The sums of $h(-4Q\ell)$ and $h(-Q\ell)$ are similar; we just explain the proof for $h(-Q\ell)$.

For a discriminant $-D < -4$, we have $h(-D) = \frac{\sqrt{D}}{\pi} L(1, \chi_{-D})$, where $\chi_{-D}(n) = \left(\frac{-D}{n}\right)$. We have $L(1, \chi_{-D}) = \sum_{n<T} \frac{\chi_{-D}(n)}{n} + O(\sqrt{D} \log(D)/T)$ by Polya–Vinogradov and partial summation. Then

$$\sideset{}{^*}\sum_{Q<X} L(1, \chi_{-Q\ell}) = \sum_{n^2<X} \frac{1}{n^2} \sideset{}{^*}\sum_{Q<X} 1 + \sum_{\substack{n<X \\ n \notin \square}} \frac{1}{n} \sideset{}{^*}\sum_{Q<X} \chi_{-Q\ell}(n) + O(\sqrt{\ell/X} \log(\ell X)),$$

where the sums over Q are restricted to squarefree Q coprime to ℓ such that $Q\ell \equiv a \bmod m$ and $Q\ell \equiv 1 \bmod 4$.

The first double sum on the right is $c_1 X + O(\sqrt{X})$ for some fixed $c_1 > 0$. For the second sum, note that $\left|\sum_{Q<X}^* \chi_{-Q\ell}(n)\right| = \left|\left(\frac{-\ell}{n}\right) \sum \left(\frac{Q}{n}\right) \mu^2(Q)\right|$. By [Zub, Lemma 6.7],

this is $O(Q^{3/5+\varepsilon}n^{1/5+\varepsilon})$, which implies the second sum above is $O(X^{4/5+\varepsilon})$. This suffices for the corresponding class number sum estimate. $\square$

By the lemma and remarks above, the $s = 0$ contribution to (6.2) is asymptotic to a finite linear combination of the form $\sum_{i\in\mathbb{Z}/8M\mathbb{Z}} c_i\sqrt{\ell/X}$. This proves Theorem 1.9, as the hypothesis $x < \frac{1}{4M} - \varepsilon$ means that only the $s = 0$ term from the trace formula contributes to (6.1) asymptotically.

6.2. **Analysis for Type II.** Next suppose $(\mathcal{N}, \mathcal{Q}) = \{(N, Q)\}$ is of Type II with Q fixed and squarefree, and N ranges over all squarefree numbers of the form $N = QM$. As before, fix k and let $\mathcal{F}$ be the family of weight k newforms of some level $N \in \mathcal{N}$. Then

$$A^{\mathcal{Q}}_{\mathcal{F}}(\ell, X; \beta) = \frac{1}{\#\mathcal{F}(X, \beta X)}\ell^{1-\frac{k}{2}} \sideset{}{'}\sum_{\frac{X}{Q}\le M\le\beta\frac{X}{Q}} \sqrt{M}\,\mathrm{tr}_{S_k^{\mathrm{new}}(QM)}\, T_\ell W_Q, \tag{6.3}$$

where the prime on the sum means M is restricted to squarefree numbers coprime to ℓQ.

Now we will analyze an analogue of (6.3) without weighting by the factor $\sqrt{M}$, and this will motivate its inclusion.

Lemma 6.2. *Let $Q \ge 1$ be squarefree. As $\ell, X \to \infty$ with ℓ prime coprime Q, we have*

$$\ell^{\frac{1-k}{2}} \left| \sideset{}{'}\sum_{X<M<\beta X} \mathrm{tr}_{S_k^{\mathrm{new}}(QM)}\, T_\ell W_Q \right| \ll \ell^{\frac{6}{5}+\varepsilon} X^{\frac{3}{5}+\varepsilon} + \delta_{k=2}\, o(\ell X), \tag{6.4}$$

where the sum is restricted to squarefree M such that $(M, Q\ell) = 1$.

Proof. Note that the $\delta_{k=2}$ term for $\mathrm{tr}_{S_k^{\mathrm{new}}(QM)}\, W_Q T_\ell$ contributes $(\ell+1)o(X)$ to the above sum of traces of $T_\ell W_Q$, using the fact the Mertens function $M(x) = \sum_{n\le x}\mu(n)$ is $o(X)$. The $\delta_{N=1}$ term in $\mathrm{tr}_{S_k^{\mathrm{new}}(QM)}\, W_Q T_\ell$ can be ignored.

For a given s such that $s^2 \le \frac{4\ell}{Q}$, the contribution to the above sum is

$$-\frac{1}{2}U_{k-2}\Big(\frac{s}{2}\sqrt{\frac{Q}{\ell}}\Big)H(\Delta) \sideset{}{'}\sum_{X<M<\beta X} \xi_\Delta(M),$$

where $\Delta = s^2Q - 4Q\ell$. Note that $U_{k-2}(\frac{s}{2}\sqrt{\frac{Q}{\ell}})$ is absolutely bounded since U_{k-2} is a polynomial and the argument is absolutely bounded. Since $|\Delta| = O(\ell)$, we have $H(\Delta) = O(\sqrt{\ell}\log\ell)$, and by [Zub, Lemma 6.7] the sum over M is $O(X^{3/5+\varepsilon}\ell^{1/5+\varepsilon})$. Summing up the $O(\sqrt{\ell})$ terms now gives the asserted bound. $\square$

Corollary 6.3. *Let $\mathcal{F}$ be the family of weight k newforms with squarefree level, and fix $\beta > 1$. As $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$ for some $x \in [0, \infty)$, the unweighted averages satisfy*

$$A^{+}_{\mathcal{F}}(\ell, X; \beta) + A^{-}_{\mathcal{F}}(\ell, X; \beta) \to 0.$$

We actually expect more cancellation than in Lemma 6.2. If $\ell, X \to \infty$ such that $\frac{\ell}{X} \to x$, then left hand side of (6.4) divided by the number of weight k newforms in that range appears to grow roughly like $\frac{\sqrt{\ell}}{X}$. This suggests the $\sqrt{M} = \sqrt{N/Q}$ weighting in (6.3).

6.3. **Analysis for Atkin–Lehner eigenspaces.** Now fix $m < r$, primes $p_1 < \cdots < p_m$, and let $\mathcal{F}, \mathcal{N}$ be as in Conjecture 1.10. Say $N = p_1 \dots p_r \in \mathcal{N}$ with $p_1 < \cdots < p_r$, and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \pm 1^r$. We view ε as the multiplicative function on divisors N such that $\varepsilon(p_i) = \varepsilon_i$. Then

$$\mathrm{tr}_{S_k^{\mathrm{new}}(N)^\varepsilon} T_\ell = 2^{-r} \sum_{Q|N} \varepsilon(Q) \, \mathrm{tr}_{S_k^{\mathrm{new}}(N)} T_\ell W_Q.$$

(See [Mar18a, Proposition 3.2] for the case of $\ell = 1$, but the proof works for general ℓ.)

For a subset $I \subset \{1, \dots, r\}$, denote by $Q_I = \prod_{i \in I} p_i$ the divisor of some $N \in \mathcal{N}$, and let $\mathcal{Q}_I$ the sequence of Q_I's as N ranges over $\mathcal{N}$. Since $\dim S_k^{\mathrm{new}}(N)^\varepsilon \approx 2^{-r} \dim S_k^{\mathrm{new}}(N) + O(1)$ by [Mar18a, Corollary 3.4], one can approximate the averages for the Atkin–Lehner eigenspace

$$A_{\mathcal{F}}^{\varepsilon}(\ell, X; \beta) \approx 2^{-r} \sum_{I \subset \{1,\dots,r\}} \varepsilon(Q_I) c_{I,X,\beta} A_{\mathcal{F}}^{\mathcal{Q}_I}(\ell, X; \beta), \quad \text{where } c_{I,X,\beta} = \sideset{}{'}\sum_{X \le N \le \beta X} \delta_{N \in \mathcal{N}} \sqrt{\frac{Q_I}{N}}.$$

Assuming Conjecture 1.8, the terms on the right should only contribute in a limit if $c_{I,X} \not\to 0$ as $X \to \infty$. Hence we expect a relation between the murmurations in Conjectures 1.8 and 1.10 of the form

$$M_{\mathcal{F}}^{\varepsilon}(x; \beta) = \sum_{\{1,\dots,m\} \subset I \subset \{1,\dots,r\}} \tilde{c}_{I,x,\beta} M_{\mathcal{F}}^{\mathcal{Q}_I}(x; \beta). \tag{6.5}$$

This justifies the expectations in Remark 1.11(2).

6.4. **Levels divisible by ℓ.** Here we indicate what happens if one includes levels $\ell \mid N$ (as is done in [Zub]) in murmurations sums. For simplicity, let us consider the averages $A_{\mathcal{F}}^{\pm}(\ell, X; \beta)$ introduced first in Section 1.3. For a form f with level N divisible by ℓ, we have $|\ell^{1-k/2} a_\ell(f)| \le 1$. Hence

$$\sum_{\mathcal{F}^{\pm}(X, \beta X)} \ell^{1-k/2} a_\ell(f) = \sum_{\mathcal{F}^{\pm}(X, \beta X)^{(\ell)}} \ell^{1-k/2} a_\ell(f) + O(\frac{X}{\ell}).$$

Assuming $\frac{\ell}{X} \to x$, the error term in this expression is $O(1)$ and will go to 0 upon dividing by $\#\mathcal{F}^{\pm}(X, \beta X)^{(\ell)}$ or $\#\mathcal{F}^{\pm}(X, \beta X)$. Since $\#\mathcal{F}^{\pm}(X, \beta X)^{(\ell)} \approx (1 - \frac{1}{\ell}) \#\mathcal{F}^{\pm}(X, \beta X)$, we see there is no asymptotic difference between working with averages over $\mathcal{F}^{\pm}(X, \beta X)^{(\ell)}$ or $\mathcal{F}^{\pm}(X, \beta X)$.

7. Quadratic twists

Let $f \in S_k(N)$, and χ be a quadratic Dirichlet character of conductor M. From [AL78, Proposition 3.1], one knows that $f \otimes \chi \in S_k(\mathrm{lcm}(N, M^2))$. In particular, twisting by χ acts on eigenforms in $S_k(N)$ if $M^2 \mid N$. Note that if $v_p(N) > 2 v_p(M)$ for all $p \mid M$, then twisting by χ acts on newforms in $S_k(N)$: if $f \in S_k(N)$ is a newform, and $g = f \otimes \chi$ had smaller level N', then necessarily $v_p(N') < v_p(N)$ for some $p \mid M$, but then $f = g \otimes \chi$ would have level which is strictly smaller than N at p.

Here we will examine when twisting by a quadratic character produces a bijection between newforms in $S_k^{\mathrm{new}}(N)^{+_q}$ and newforms in $S_k^{\mathrm{new}}(N)^{-_q}$. For simplicity we will restrict to the case that $v_p(N) > 2 v_p(M)$ for all $p \mid M$, which is generically necessary. (If this is not satisfied, there will be some non-minimal forms where twisting by χ strictly lowers the level, except in the small parameter cases where all relevant lower level spaces are 0-dimensional.)

Say π_q is the irreducible admissible representation of $\mathrm{PGL}_2(\mathbb{Q}_q)$ associated to a newform $f \in S_k(q^r M)$, where $(M, q) = 1$ and $r \geq 1$. Then r is the conductor of π_q. If π_q is supercuspidal, there are 3 distinct possibilities: (i) it is dihedrally induced from a ramified quadratic extension $E_q/\mathbb{Q}_q$; (ii) it is dihedrally induced from the unramified quadratic extension of $\mathbb{Q}_q$; or (iii) it is not dihedrally induced. We respectively call these cases: (i) ramified supercuspidal; (ii) unramified supercuspidal; and (iii) exceptional supercuspidal. The exceptional case only happens when $q = 2$.

If $r = 1$, then π_q is an unramified twist of the Steinberg representation. If $r = 2$, then π_q can be a ramified principal series, ramified twist of Steinberg, or unramified supercuspidal. If $r \geq 3$ is odd, then π_q is ramified supercuspidal or exceptional supercuspidal (the latter only happens when $q = 2$ and $r = 3, 7$). If $r \geq 4$ is even, either π_q is a ramified principal series representation, unramified supercuspidal, or exceptional supercuspidal (the latter only occurs when $q = 2$ and $r = 4, 6$).

For a quadratic Dirichlet character χ, denote by $\kappa(\pi_q, \chi)$ the change in the W_q-eigenvalue of f upon twisting by χ, i.e., the ratio of the W_q-eigenvalues of f and $f \otimes \chi$. This only depends on π_q, and the calculation of $\kappa(\pi_q, \chi)$ is given in [Pac13] (see also [AL70, AL78] for a more classical perspective in special cases).

Since any quadratic χ is a product of quadratic characters of prime-power conductor, we may reduce to the case of twisting by characters ramified at a single finite prime p. For an odd prime p, let χ_p denote the quadratic character of conductor p, which corresponds to the quadratic extension $\mathbb{Q}(\sqrt{p^*})$ where $p^* = \left(\frac{-1}{p}\right)p$. That is, $\chi_p(n) = \left(\frac{p^*}{n}\right)$. For $j \in \{-1, \pm 2\}$, let χ_j be the quadratic character associated to $\mathbb{Q}(\sqrt{j})$. Then χ_{-1} has conductor 4 and $\chi_{\pm 2}$ has conductor 8.

7.1. **Twisting at q.** First we state $\kappa(\pi_q, \chi_q)$ when q is odd. Since we are interested in the case where twisting by χ_q acts on the newforms in $S_k(q^r M)$, we may assume the conductor of π_q is $r \geq 3$.

If π_q is a ramified principal series, then $\kappa(\pi_q, \chi_q) = \left(\frac{-1}{q}\right)$.

If π_q is an unramified supercuspidal (so r is even), then $\kappa(\pi_q, \chi_q) = -\left(\frac{-1}{q}\right)$.

If π_q is a ramified supercuspidal (so r is odd) induced from $E_q/\mathbb{Q}_q$, then $\kappa(\pi_q, \chi_q) = \pm 1)$, where the sign is $+1$ if $E_q = \mathbb{Q}_q(\sqrt{q^*})$ and -1 if $E_p = \mathbb{Q}_p(\sqrt{-q^*})$.

Thus for any q odd and $r \geq 3$, twisting by χ_q never flips the Atkin–Lehner sign of every kind of representation π_q of conductor r. In particular, twisting by χ_q does not force $\Delta_k(q^r, M) = 0$ (at least assuming that $\dim S_k^{\mathrm{new}}(q^r M)$ is sufficiently large so all possible local representations occur).

When $q = 2$, the situation is similar. If $\chi \in \{\chi_{-1}, \chi_{\pm 2}\}$, and $r \geq 5$, one may see from the calculations of $\kappa(\pi_q, \chi)$ in [Pac13, Theorem 4.2] that twisting by χ will not flip the Atkin–Lehner sign of each kind of representation $\mathrm{PGL}(\mathbb{Q}_2)$ of conductor r.

7.2. **Twisting away from q.** Next we consider twisting by a quadratic character ramified only at a prime $p \neq q$.

First suppose p is odd and $p \neq q$. Then $\kappa(\pi_q, \chi_p) = \left(\frac{q}{p}\right)^r$ for any π_q of conductor r.

Next let $\chi \in \{\chi_{-1}, \chi_{\pm 2}\}$. Then for q odd, we have $\kappa(\pi_q, \chi) = \chi(q)^r$ for any π_q of conductor r. In particular, if r is odd then $\kappa(\pi_q, \chi_{-1}) = -1$ if $q \equiv 3 \bmod 4$ and $\kappa(\pi_q, \chi_{-1}) = -1$ if $q \equiv 5 \bmod 8$.

Proposition 7.1. *Suppose $N = q^r M$, with r odd, $(q, M) = 1$, and one of the following holds:*

(1) there exists an odd p such that $p^3 \mid N$ and $\left(\frac{q}{p}\right) = -1$;
(2) $2^5 \mid N$ and $q \equiv 3 \bmod 4$;
(3) $2^7 \mid N$ and $q \equiv 5 \bmod 8$.

Then $f \mapsto f \otimes \chi$ defines a bijection of newforms in $S_k^{\mathrm{new}}(N)^{+_q}$ with $S_k^{\mathrm{new}}(N)^{-_q}$, where we can take $\chi = \chi_p$ in case (1), $\chi = \chi_{-1}$ in case (2), $\chi = \chi_{\pm 2}$ in case (3).

Note that when the hypotheses of this proposition hold, one also gets that $\mathrm{tr}_{S_k^{\mathrm{new}}(N)} T_\ell W_q = 0$ for ℓ such that $\chi(\ell) = 1$. Moreover, since $f \equiv f \otimes \chi \bmod 2$, each newform in $S_k^{\mathrm{new}}(N)^{+_q}$ is congruent mod 2 to a newform in $S_k^{\mathrm{new}}(N)^{-_q}$, and vice versa.

Appendix A. Errata for "Rank bias for elliptic curves mod p" by Kimball Martin and Thomas Pharis

Here we correct a sign error when $k \equiv 0 \bmod 4$ in Section 2 of the published article [MP22]. This has no effect on the rest of the paper.

The following corrections should be made to [MP22]:

(1) p. 710, bottom (Section 1A): the phrase "however the signs for $k \equiv 0 \bmod 4$ are opposite to those for $k \equiv 2 \bmod 4$" should be removed.

(2) p. 717: The conclusion of Proposition 2.2 should read

$$\left| \mathrm{tr}_{S_k^{\mathrm{new}}(N)^{\pm}} T_n \mp \frac{1}{4} n^{\frac{k-2}{2}} H(4nN) \right| < \left(2^{\omega(N)} (4n)^{\frac{k}{2}} + \delta_{k,2} \right) \sigma_1(n).$$

(3) p. 717, proof of Proposition 2.2: $p_k(0, n) = (-n)^{(k-2)/2}$, not $n^{(k-2)/2}$, so (2-2) should read

$$\mathrm{tr}_{S_k(N)} T_n W_N = -\frac{1}{2} (-n)^{\frac{k-2}{2}} H(4nN) + \delta_{k,2} \sigma_1(n). \tag{A.1}$$

Corresponding sign changes should be made throughout of proofs of Proposition 2.2 and Corollary 2.3.

(4) p. 717: The conclusion of Proposition 2.2 should read

$$\left| \mathrm{tr}_{S_k^{\mathrm{new}}(N)^{\pm}} T_n \mp \frac{1}{4} n^{\frac{k-2}{2}} H(4nN) \right| < \left(2^{\omega(N)} (4n)^{\frac{k}{2}} + \delta_{k,2} \right) \sigma_1(n).$$

(5) p. 718: The conclusion of Corollary 2.3 should read

$$N^{\frac{1}{2} - \epsilon} \ll \pm \mathrm{tr}_{S_k^{\mathrm{new}}(N)^{\pm}} T_n \ll N^{\frac{1}{2}} \log N.$$

(6) p. 718, bottom: the phrase "when $k \equiv 2 \bmod 4$, and approximately like $\mp\sqrt{N}$ when $k \equiv 0 \bmod 4$" should be removed.

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Department of Mathematics, University of Oklahoma, Norman, OK 73019 USA
Email address: kimball.martin@ou.edu

Department of Mathematics · International Research and Education Center, Graduate School of Science, Osaka Metropolitan University, Osaka 558-8585, Japan
Email address: kimball@omu.ac.jp