REPRESENTATIONS OF S_3, A_4 AND S_4

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Preliminaries

Let us recall *Mackey's criterion*, which we will use to classify the represention of S_3, A_4 and S_4. Let H be a subgroup of a finite group G and let H_s be the subgroup $sHs^{-1} \cap H$ of H. Let (ρ, W) be a representation of H. Define two representations ρ^s and $Res_s(\rho)$ on H_s by $\rho^s(x) = \rho(s^{-1}xs)$ and $\mathrm{Res_s}(\rho) = \rho|_{\mathrm{H_s}}$. We can use these representations to determine if the induced representation $V = \mathrm{Ind_H^G(W)}$ is irreducible.

Mackey's Criterion. *With notation as above, V is irreducible if and only if both W is irreducible and for all $s \in G - H$ the representations ρ^s and $\mathrm{Res_s}\rho$ of H_s have no irreducible components in common.*

Corollary 1. *Suppose H is normal in G. (In particular, $H_s = H$.) Then V is irreducible if and only if ρ is irreducible and not isomorphic to ρ^s for all $s \in G - H$. Note if* $\deg \rho = 1$ *the latter hypothesis is equivalent to $\rho \neq \rho^s$ for all $s \in G - H$.*

Representations of S_3

Since S_3 is non-abelian, not all of its irreducible representations have degree 1. The only way to write $|S_3| = 6$ as a sum of squares not all 1 is $6 = 1^2 + 1^2 + 2^2$. Thus S_3 has three (inequivalent) irreducible representations: two of degree 1 and one of degree 2. Write

$$S_3 = < \sigma, \tau : \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} > .$$

If χ is a character of S_3, then $\chi(\sigma)^3 = \chi(\tau)^2 = 1$ and $\chi(\tau)\chi(\sigma)\chi(\tau) = \chi(\sigma)^{-1}$. The condition $\chi(\tau)^2 = 1$ makes the latter equation equivalent to $\chi(\sigma) = \chi(\sigma)^{-1}$. Combining this with $\chi(\sigma)^3 = 1$ forces $\chi(\sigma) = 1$. Hence the two irreducible representations of degree 1 are the trivial character χ_0 and the character χ obtained by sending σ to 1 and τ to -1.

Now let us see if we can obtain the irreducible representation of degree 2 as an induced representation from a degree one representation of a subgroup H. For the degree to be two, the subgroup will have to have index 2 in S_3. So take $H = < \sigma >$. The characters of H are ψ_0, ψ, ψ^2 where ψ_0 is the trivial character, $\psi(\sigma^j) = e^{2\pi i j/3}$. So $\rho = \mathrm{Ind_H^{S_3}}\psi$ is a representation of degree 2 of S_3 and we will show it is irreducible by Mackey's Criterion. Clearly H is normal so we may use Cor 1. We can write $s \in S_3 - H$ as $\tau\sigma^i$ and $x \in H$ as σ^j. Then

$$\psi^s(x) = \psi(s^{-1}xs) = \psi(\sigma^{-i}\tau\sigma^j\tau\sigma^i) = \psi(\sigma^{-i}\sigma^{-j}\sigma^i) = \psi(\sigma^{-j}) = \psi(x^{-1}) = \psi^2(x).$$

Since $\psi^2 \neq \psi$, ρ is irreducible as claimed. Similarly $\mathrm{Ind_H^{S_3}}\psi^2$ is also an irreducible representation of S_3 of degree 2, which means it must be equivalent to ρ.

Representations of A_4

Consider A_4 as the set of even permutations of the set $\{1, 2, 3, 4\}$. Let $x = (12)(34), y = (13)(24)$ and $z = (14)(23)$ so $A = \{1, x, y, z\}$ is a subgroup of A_4 of order 4. The only other elements of A_4 are the eight 3-cycles of the form (abc). Let $t = (123)$ and observe that all the 3-cycles are the elements of $tA \cup t^2A$. So $A_4 = HA$ where $H = < t >$. Also note that $txt^{-1} = z, tyt^{-1} = x, tzt^{-1} = y$, so A is normal in A_4. Thus A_4 is a semidirect product $H * A$.

The group A_4 has four conjugacy classes: $\{1\}, A - \{1\}, tA$, and t^2A, and thus has four irreducible representations. Tne only way to write 12 as a sum of four squares is $12 = 1^2 + 1^2 + 1^2 + 3^2$, so A_4 has three representations of degree 1 and one irreducible representation of degree 3. The three degree 1 representations χ_0, χ_1, χ_2 can be obtained from the three characters of H via $\chi_j(ha) = \chi_j(h)$ for all $h \in H, a \in A$. (Here χ_j

is defined on H by $\chi_j(t^k) = e^{2\pi ijk/3}$.) Clearly the χ_j's so defined on A_4 are class functions and it is easy to check that the χ_j's are indeed homomorphisms.

We would like to be able to construct the degree 3 representation as an induced representation of a subgroup. The subgroup will have to have index 3 in A_4 so let us consider the induced representations from A. Let ψ be a nontrivial irreducible character of A (so it sends two of $\{x, y, z\}$ to -1 and the other to 1). We want to see if $\rho = \mathrm{Ind}_{\mathrm{A}}^{\mathrm{A}_4}\psi$ is irreducible. For each $s \in A_4 - A$, s does not act trivially on A by conjugation so $\psi^s \neq \psi$. So by Cor 1, ρ is the irreducible representation of A of degree 3. Notice we get the degree 3 representation (up to equivalence) of A_4 by inducing from any nontrivial character of A.

Representations of S_4

Consider S_4 as the permutation group of {1,2,3,4}. It has order 24 and we will exhibit two irreducible representations of degree 1, one of degree 2 and two of degree 3. As $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$, this comprises all the irreducible representations of S_4. Let A be as above, so it is a normal subgroup of S_4 of order 4. Regard $H = S_3 \subseteq S_4$ as the permutations which fix $\{4\}$. Then S_4 is the semidirect product $S_3 * A$. Notice that, as with A_4 we can obtain irreducible representations of S_4 from the irreducible representations of H via $\rho(ha) = \rho(h)$ for all $h \in H, a \in A$ and ρ an irreducible representation of H. The irreducible representations of S_3 were determined above. Denote them by χ_0 (the trivial character), χ_1 (the nontrivial character of degree 1 of S_3) and ρ_2 (the irreducible representation of degree 2).

Let P be a Sylow 2-subgroup of S_4, i.e. P has order 8. Note that there is more than one subgroup of order 8 so, being a Sylow 2-subgroup, P is not normal is S_4. Since $[S_4 : P] = 3$ we can induce degree 1 representations of P to get degree 3 representations of S_4. We can write P as a semidirect product $K * A$, where K has order 2. Write $A = \{1, x, y, z\}$. The group K acts by conjugation on A and this action will fix precisely one nontrivial element of A, say x. Let χ be the the irreducible character of A such that $\chi(1) = \chi(x) = 1$ and $\chi(y) = \chi(z) = -1$. Then χ extends to a degree one character on P by $\chi(ka) = \chi(a)$ for all $k \in K, a \in A$.

Given a degree one character of K, we can also obtain a degree one character on P by composing it with the natural projection from P to K. Call these characters ψ_0, ψ_1. Then the products $\chi\psi_i$ are also degree one (= irreducible) characters of P. Let $\rho_i = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{S}_4}\chi\psi_{\mathrm{i}}$ for $i = 1, 2$. We claim that ρ_0, ρ_1 are irreducible and inequivalent, thus giving the two irreducible representations of degree three.

To show they are inequivalent, consider the representation space W for ρ_i. Let $W' = \{x \in W | \rho_i(a)x = \chi(a)x \forall a \in A\}$. Note W' is stable under K and the representation of K in W' is isomorphic to ψ_i. Thus ρ_i determines ψ_i.

We use Mackey's criterion to show ρ_i is irreducible. Let $s \in S_4 - P$ and $P_s = sPs^{-1} \cap P$. We want to show $\mathrm{Res}_{\mathrm{s}}(\rho_{\mathrm{i}})$ and ρ_i^s have no irreducible components in common. It suffices to show this after restricting to the subgroup $A \leq P_s$. But they restrict to distinct degree one representations of A, so they have no components in common. Thus ρ_i is irreducible.

Note this method of constructing irreducible induced representations is a special case of a more general method for semidirect products by abelian normal subgroups. See Serre, Section 8.2.

References

1. Serre, J.-P. Linear Representations of Finite Groups. Springer-Verlag, 1977.