

RATIONALITY FIELDS OF CM MODULAR FORMS

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ABSTRACT. We formulate a conjecture on the finitude of rationality fields (i.e., Fourier coefficient fields) of newforms of bounded degree, and prove this for CM forms assuming a generalized Riemann hypothesis. Then we explicitly determine what quadratic and cubic rationality fields occur for weight 2 CM forms, which is related to classifications of CM abelian varieties of $GL(2)$ type. Several of these fields do not appear in the existing tables of newforms in the LMFDB.

1. INTRODUCTION

The motivation for this paper begins with the following conjecture. For a newform $f = \sum a_n q^n \in S_k(N) = S_k(\Gamma_0(N))$, the rationality field $K_f = \mathbb{Q}((a_n)_n)$ is the number field generated by the Fourier coefficients of f .

Conjecture 1. *Fix $k \geq 2$ even and $d \geq 1$. There are only finitely many number fields K of degree d such that $K = K_f$ for some weight k newform f with trivial nebentypus.*

To our knowledge, this conjecture has not been previously formulated in the literature, but it is suggested by existing ideas and conjectures.

First, Coleman’s conjecture says that, for a fixed d , the set of isomorphism classes of endomorphism algebras $\text{End}_{\mathbb{Q}}^0(A)$ of d -dimensional simple abelian varieties A/\mathbb{Q} is finite (see [BFGR06]). In particular, if A_f denotes the (simple) abelian variety attached to a weight 2 newform f à la Shimura, then $\text{End}_{\mathbb{Q}}^0(A) \simeq K_f$ by [Rib04]. Hence Coleman’s conjecture implies **Conjecture 1** when $k = 2$.

Next, [Mar21, Conjecture B+] predicts that there are only finitely many newforms of squarefree level (and trivial nebentypus) with a degree d rationality field for fixed $k \geq 6$ and $d \geq 1$. The heuristics behind the conjecture also suggest that there are only finitely many quadratic twist classes of weight k non-CM newforms with rationality field having degree d for large k . This would imply **Conjecture 1** for $k \geq 6$ when restricting to non-CM newforms. (**Theorem 1** below addresses the CM case.)

Further, [CM] suggests that one reason **Conjecture 1** may hold when $k = 2$ is that there may only be finitely many related moduli spaces having non-cuspidal rational points for a fixed d . (When $d = 1$, this means that only finitely many modular curves $X_0(N)$ have non-cuspidal rational points, which is known; see [Maz78, Ken82].) In higher weight, newforms do not correspond to abelian varieties but motives. Philosophically it seems reasonable to posit an analogous geometric finiteness statement about motives which would imply **Conjecture 1** for all k .

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At present, there is no clear strategy to tackle [Conjecture 1](#), even for $k = 2$. Our first result proves this conjecture when one restricts to CM forms, subject to the Extended Riemann Hypothesis (ERH) for quadratic Dirichlet characters.

Theorem 1. *Fix $k \geq 2$ and $d \geq 1$. There are only finitely many number fields K of degree d such that $K = K_f$ for some weight k newform f with CM and trivial nebentypus, assuming ERH. If $k = 2$ and $d \leq 6$, or if $(k - 1)d = 2^n 3^\epsilon$ with $n \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$, then this holds unconditionally.*

Such an f as in the theorem must have CM by an imaginary quadratic field E with (class group) exponent $n = (k - 1)d$ (see [Corollary 3.2](#)). The assumption of ERH guarantees that there are only finitely many imaginary quadratic fields of exponent n [[BK72](#)]. The latter assertion is known unconditionally when $n \leq 6$, or n is a power of 2, or n is 3 times a power of 2 [[HB08](#)]. Then we prove [Theorem 1](#) by showing an analogous finiteness statement for the collection of value fields of bounded degree of Grössencharacters of a fixed field E (see [Corollary 3.6](#)).

Now it is natural to ask, for a given k and d , what rationality fields actually occur? This question is considered in [[CM](#)] (which focuses on weight 2 non-CM forms), where for $k = 2$ it is related to the existence of rational points on suitable Hilbert modular varieties. However even for $k = 2$ and $d = 2$, it is not clear how to even guess which rationality fields might occur. (The only option for $d = 1$ is $K_f = \mathbb{Q}$, and it easy to see it occurs for each k .)

Under the assumption of ERH (and in some cases a weaker hypothesis on non-existence of Siegel zeroes), [[EKN20](#)] enumerates all E with exponent at most 5 or exponent 8. It is unconditionally known that there is at most one other E for exponent 2, 4 or 8 [[Wei73](#), [EKN20](#)]. By determining the possible rationality fields for a given E in the conditionally complete list of exponent ≤ 3 fields, we deduce the following.

Theorem 2. *Assuming ERH, the complete set of degree 2 and 3 rationality fields K of weight 2 CM forms are the 23 quadratic fields and 18 cubic fields with discriminant Δ_K listed in [Tables 1.1](#) and [1.2](#), respectively.*

In [Tables 1.1](#) and [1.2](#), along with each Δ_K , we list the discriminants Δ_E of all E such that $K = K_f$ for some newform f with CM by E (subject to ERH). In all cases listed, the degree and discriminant of K determine K up to isomorphism. The first two cubic fields K appearing in [Table 1.2](#) are $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$.

In [Table 1.1](#), the E 's which appear are precisely the imaginary quadratic fields of class number 1 or 2. When $h_E = 2$ the corresponding K is the real quadratic field such that $L = EK$ is the Hilbert class field of E (though we do not have a direct proof of this fact). In [Table 1.2](#) the E 's which appear are the imaginary quadratic fields of class number 3, and the two class number one fields $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$. but when $h_E = 3$, the Hilbert class field of E has no nontrivial totally real subfields so cannot contain K .

Our strategy for [Theorem 2](#) is, for each E with exponent dividing d , to first determine a potential list of degree d rationality fields K using necessary conditions on relevant Grössencharacters of E . Then for each such K we either construct a Grössencharacter yielding this K or prove this is impossible by contradiction. For this, we consider Grössencharacters in 3 cases: (i) when E has class number 1 ([Section 5](#)), (ii) when an associated modulus character is quadratic and $h_E > 1$ ([Section 6](#)), and (iii) when the associated modular character has higher order and $h_E > 1$ ([Section 7](#)).

Δ_K	Δ_E	Δ_K	Δ_E	Δ_K	Δ_E
5	-15, -20, -35, -40, -115, -235	29	-232	89	-267
8	-4, -8, -24, -88	33	-11	129	-43
12	-3, -4	37	-148	172	-43
13	-52, -91, -403	41	-123	201	-67
17	-51, -187	44	-11	268	-67
21	-7	57	-19	489	-163
24	-8	61	-427	652	-163
28	-7	76	-19		

TABLE 1.1. Quadratic rationality fields K for weight 2 forms with CM by E

Let us discuss how the rationality fields in [Theorem 2](#) compare to known tables of low degree rationality fields. The LMFDB¹ tabulates low degree rationality fields for all weight 2 newforms of level at most 50000, thanks to the recent work [\[Boo24\]](#).

There are 59 real quadratic fields K which occur in this range. (The first missing fundamental discriminant is 77, and the largest occurring discriminant is 356.) The two largest discriminants from [Table 1.1](#), namely 489 and 652, do not occur in the LMFDB data. (Our constructions in [Section 5.2](#) produce CM forms f with these rationality fields in levels $106276 = 2^2 \cdot 163^2$ and $239121 = 3^2 \cdot 163^2$.) We also note that there is exactly one newform f in this range with rationality field $K = \mathbb{Q}(\sqrt{43})$ (i.e., $\Delta_K = 172$), and similarly for $K = \mathbb{Q}(\sqrt{67})$ (i.e., $\Delta_K = 268$). These forms occur in level $3^2 \cdot 43^2$ and $3^2 \cdot 67^2$, and our constructions in [Section 5.2](#) show that these are CM forms. (CM data for newforms in the LMFDB of level over 10000 has not been computed.)

There are 161 cubic rationality fields in the LMFDB data, with the largest field having discriminant 20613. Six of the rationality fields K from [Table 1.2](#) do not appear in the LMFDB data, namely those K with $\Delta_K = 7641$ or $\Delta_K \geq 10233$. Construction of CM forms with these 6 rationality fields occur in level Δ_E^2 , which ranges from 80089 for $\Delta_K = 7641$ to 822649 for $\Delta_K = 24489$ (see [Section 6.2](#)).

Remark 1. We expect that with more work [Theorem 2](#) could be extended to $d = 4, 5, 8$ for weight 2 newforms. To get complete lists of rationality fields of CM forms for other cases d or k , even subject to ERH, would require extending [\[EKN20\]](#) to other exponents.

One reason we only attempted to treat $d = 2, 3$ is that for $d = 4$ one needs to analyze all exponent 4 fields. Assuming ERH, there are many more fields of exponent 4 (203 fields) than exponent 2 (56 fields) or exponent 3 (17 fields). Furthermore, as will be clear from the proof, there are fewer situations to consider when $(k - 1)d$ is prime.

Finally, we note the following consequence on the existence of abelian varieties with special properties. Recall from [\[Shi71\]](#) that, given a newform $f \in S_2(N)$ with CM by E , the associated abelian variety A_f is \mathbb{Q} -isogenous to \mathcal{E}^d for some elliptic curve \mathcal{E}/\mathbb{Q} with CM by E , where $d = [K_f : \mathbb{Q}]$.

Corollary 3. *For each pair (E, K) occurring in [Table 1.1](#) and [Table 1.2](#), there is simple abelian variety A/\mathbb{Q} of dimension $d = [K : \mathbb{Q}]$ with \mathbb{Q} -endomorphism algebra K such that A is \mathbb{Q} -isogenous to a power of an elliptic curve \mathcal{E}/\mathbb{Q} with CM by E .*

¹<https://www.lmfdb.org/>

Δ_K	Δ_E	Δ_K	Δ_E	Δ_K	Δ_E
49	-7	1593	-59	8289	-307
81	-3	1929	-643	10233	-379
321	-107	2241	-83	13473	-499
621	-23	3753	-139	14769	-547
837	-31	5697	-211	23841	-883
993	-331	7641	-283	24489	-907

TABLE 1.2. Cubic rationality fields K for weight 2 newforms with CM by E

In particular, this gives some constructive information about the question: for which totally real fields K does there exist a d -dimensional simple abelian variety over \mathbb{Q} with real multiplication (defined over \mathbb{Q}) by an order in K ?

Remark 2. A related problem is, for a number field F , to classify CM elliptic curves \mathcal{E} with minimal field of definition F , i.e., $F = \mathbb{Q}(j(\mathcal{E}))$. For F quadratic, there exist such \mathcal{E} only for 14 real quadratic fields, namely the 11 real quadratic fields with discriminant $\Delta < 40$, $\mathbb{Q}(\sqrt{41})$, $\mathbb{Q}(\sqrt{61})$ and $\mathbb{Q}(\sqrt{89})$. Curves for such fields are given in [GJLRY25].

Note that these 14 real quadratic fields F are a proper subset of the 23 real quadratic fields K . When \mathcal{E}/K has CM by the maximal order of a class number two field E , this follows from the coincidence in Table 1.1 between EK and the Hilbert class field mentioned above. In general, one might wonder to what extent there is overlap between the CM newforms f with quadratic rationality field K and CM elliptic curves \mathcal{E} with minimal field of definition $F = K$.

According to the LMFDB, the first 3 weight 2 CM newforms f when ordered by level (LMFDB labels 225.2.a.f, 256.2.a.e, 441.2.a.h) have respective rationality fields $K = \mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{7})$ and all correspond to elliptic curves \mathcal{E} with minimal field of definition $F = K$. Hence there is some overlap, but this pattern does not persist – the next CM form f (LMFDB label 484.2.a.d) has rationality field $K = \mathbb{Q}(\sqrt{33})$, but one can check that its base change to K does not have rational Fourier coefficients, so A_f is not K -isogenous to an elliptic curve defined over $F = K$. Conversely, there are CM elliptic curves over quadratic fields F , which do not correspond to modular forms over \mathbb{Q} , e.g., the elliptic curve with LMFDB label 4091.1-k1 is defined over $F = \mathbb{Q}(\sqrt{5})$ and the associated Hilbert modular form is not a base change from \mathbb{Q} .

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2. CM FORMS

In this section, we set some notation, recall basic facts about Grössencharacters and CM forms, and relate the rationality fields of CM forms to the fields of values of Grössencharacters.

2.1. Notation. For a field F , let μ_F be the group of roots of unity in F . For a finite-degree field extension K/F , denote by $\mathcal{N} = \mathcal{N}_{K/F}$ the norm map from K to F . If F is a number field, Δ_F denotes its discriminant, \mathfrak{o}_F its ring of integers, $\text{Cl}(F)$ its ideal class group, and $h_F = \#\text{Cl}(F)$.

For $n \geq 1$, let ζ_n be a primitive n -th root of unity, and write $\mu_n = \langle \zeta_n \rangle$. We denote the cyclic group of order n by C_n .

Throughout, E/\mathbb{Q} will be an imaginary quadratic field, p will denote a prime in \mathbb{N} and \mathfrak{p} will denote a prime of E above p . Further, \mathfrak{m} will denote a nonzero integral ideal of E .

For $\alpha \in E^\times$, write $\alpha \equiv 1 \pmod{\mathfrak{m}}$ if $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{m})$ for each prime $\mathfrak{p} \mid \mathfrak{m}$. (Some authors write $\alpha \equiv 1 \pmod{\times \mathfrak{m}}$.) Let $E^{\mathfrak{m}} = \{\alpha \in E^\times : v_{\mathfrak{p}}(\alpha) = 0 \text{ for all } \mathfrak{p} \mid \mathfrak{m}\}$ and $E_1^{\mathfrak{m}} = \{\alpha \in E^\times : \alpha \equiv 1 \pmod{\mathfrak{m}}\}$. Denote by $J^{\mathfrak{m}} = J^{\mathfrak{m}}(E)$ the group of (invertible) fractional ideals of E relatively prime to \mathfrak{m} , $P^{\mathfrak{m}} = P^{\mathfrak{m}}(E)$ the subgroup of principal ideals, and $P_1^{\mathfrak{m}} = P_1^{\mathfrak{m}}(E) = \{\alpha \mathfrak{o}_E : \alpha \in E_1^{\mathfrak{m}}\}$ the subgroup of principal ideals which are $1 \pmod{\mathfrak{m}}$. The associated ray class group is $\text{Cl}^{\mathfrak{m}}(E) = J^{\mathfrak{m}}(E)/P_1^{\mathfrak{m}}(E)$.

2.2. Grössencharacters and CM forms. Fix an imaginary quadratic field E/\mathbb{Q} , and a nonzero integral ideal \mathfrak{m} of E . Let ψ be a Grössencharacter of E with conductor \mathfrak{m} and weight $\ell \in \mathbb{Z}_{\geq 0}$, which means $\psi : J^{\mathfrak{m}} \rightarrow \mathbb{C}^\times$ is a character such that

$$\psi(\alpha \mathfrak{o}_E) = \alpha^\ell \quad \text{if } \alpha \equiv 1 \pmod{\mathfrak{m}}.$$

Given \mathfrak{m} and ℓ , there exists such a ψ if and only if $\zeta^\ell = 1$ for each $\zeta \in \mathfrak{o}_E^\times \cap E_1^{\mathfrak{m}}$. Note that $\mathfrak{o}_E^\times \cap E_1^{\mathfrak{m}}$ is only nontrivial if (i) $2 \in \mathfrak{m}$ or (ii) $\Delta_E = -3$ and $\mathcal{N}(\mathfrak{m}) \in \{1, 3\}$.

The map $E^{\mathfrak{m}}/E_1^{\mathfrak{m}} \simeq (\mathfrak{o}_E/\mathfrak{m})^\times \rightarrow P^{\mathfrak{m}}/P_1^{\mathfrak{m}}$ is surjective with kernel isomorphic to $\mathfrak{o}_E^\times \cap E_1^{\mathfrak{m}}$. We often identify $E^{\mathfrak{m}}/E_1^{\mathfrak{m}}$ with $(\mathfrak{o}_E/\mathfrak{m})^\times$. Then ψ gives rise to a character $\eta_\psi : (\mathfrak{o}_E/\mathfrak{m})^\times \rightarrow \mathbb{C}^\times$ via

$$\eta_\psi(\alpha) = \alpha^{-\ell} \psi(\alpha \mathfrak{o}_E),$$

which we call the *modulus character* of ψ . Replacing α by $u\alpha$ shows that $\eta_\psi(u) = u^{-\ell}$ for any $u \in \mathfrak{o}_E^\times = \mu_E$. Further, η_ψ is trivial on $\mathfrak{o}_E^\times \cap E_1^{\mathfrak{m}}$.

Extend ψ to be 0 on integral ideals not coprime to \mathfrak{m} . Then the following result can be found in [Shi71, Lemma 3] and [Rib77, Section 3]. [Hecke, Shimura] Define

$$f_\psi = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N\mathfrak{a}},$$

where \mathfrak{a} runs over integral ideals of E . Let $\eta_\psi^\mathbb{Z}$ be the Dirichlet character of $\mathbb{Z} \bmod M = \mathcal{N}\mathfrak{m}$ given by composing η_ψ with the natural map $\mathbb{Z}/M\mathbb{Z} \rightarrow \mathfrak{o}_E/\mathfrak{m}$. Then

$$f_\psi \in S_{\ell+1}(|\Delta_E| M, \chi_E \eta_\psi^\mathbb{Z}).$$

If ψ is primitive (see Section 4), then f_ψ is a newform.

In particular, we have the property $f_\psi \otimes \chi_E = f_\psi$, which means that f_ψ has complex multiplication (CM) by E . Further, any newform with CM by E is of the form f_ψ for some such ψ ([Rib77, Theorem 4.5]).

We are most interested in the case that ℓ is odd and $\eta_\psi^\mathbb{Z} = \chi_E$ (as Dirichlet characters mod $|\Delta_E| M$), which corresponds to f_ψ having even weight and trivial nebentypus. This will imply that $\psi(p \mathfrak{o}_E) = \left(\frac{\Delta_E}{p}\right) p^\ell$ when $p \nmid \Delta_E$. Further, the conductor of χ_E will have to divide that of $\eta_\psi^\mathbb{Z}$, which means that $\Delta_E \mid M$ (see Lemma 4.3 for a more precise condition). The condition $\eta_\psi(-1) = \chi_E(-1) = -1$ implies that $-1 \notin \mathfrak{o}_E^\times \cap E_1^{\mathfrak{m}}$, and thus $2 \in \mathfrak{m}$ is impossible.

2.3. Rationality fields and value fields. Now we connect rationality fields of CM forms (with trivial nebentypus) with the values of Grössencharacters. By the *value field* L_ψ of a Grössencharacter ψ , we mean the subfield of \mathbb{C} generated by $\text{Im } \psi$. Then L_ψ is a number field containing E , as it is generated over E by its values on a set of representatives for the ray class group $\text{Cl}^{\mathfrak{m}}(E)$.

Lemma 2.1. *Let $f = f_\psi \in S_k(N)$ with rationality field $K = K_f$, and let $L = L_\psi$ be the value field of ψ . Then $L = EK$.*

Proof. Note that L is generated by the values $\psi(\mathfrak{p})$, and K is generated by the values $\psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$, where in both cases \mathfrak{p} ranges over prime ideals in \mathfrak{o}_E .

Say \mathfrak{p} is a prime above $p \in \mathbb{Z}$. We know that $\psi(\mathfrak{p}) \in \mathbb{Z}$ when p is inert or ramified in E , so it suffices to consider split primes $p = \mathfrak{p}\bar{\mathfrak{p}}$.

Since $\psi(\bar{\mathfrak{p}}) = \psi(p\mathfrak{o}_E)/\psi(\mathfrak{p}) = p^\ell/\psi(\mathfrak{p})$, we see that $\beta = \psi(\mathfrak{p})$ satisfies a quadratic polynomial over K , and thus β lies in a quadratic extension of K .

Say \mathfrak{p} has order n in $\text{Cl}(E)$, and write $\mathfrak{p}^n = \varpi\mathfrak{o}_E$ for some irreducible $\varpi \in E$. Let r be the multiplicative order of $\varpi \bmod \mathfrak{m}$. Then $\beta^{nr} = \psi(\mathfrak{p}^n)^r = \varpi^{r\ell}$, so $\beta^n = \psi(\mathfrak{p}^n) = \zeta\varpi^\ell$ for some $\zeta \in \mu_r$. Now any positive power of ϖ lies in $E - \mathbb{Q}$ (or else \mathfrak{p} would ramify over p), so some positive power of β generates E . However, since β was quadratic over K and E must be disjoint from the totally real field K , one deduces $\beta \in EK$. This implies $L \subseteq EK$.

For the converse, note that $L \supseteq K$ is clear. Now by the previous paragraph, some power of $\beta = \psi(\mathfrak{p})$ generates E and thus $L \supseteq EK$. \square

Thus we can characterize the rationality field K of $f_\psi \in S_k(N)$ as the unique totally real index 2 subfield of the value field L of ψ . Hence to study rationality fields of CM forms with trivial nebentypus, it suffices to study value fields of Grössencharacters.

3. VALUE FIELDS OF GRÖSSENCHARACTERS

Here we prove some basic properties about value fields of Grössencharacters and deduce [Theorem 1](#). Let ψ be a Grössencharacter of E of conductor \mathfrak{m} and weight ℓ . Denote by $L = L_\psi$ its value field, and set $d = [L : E]$.

3.1. Ideal class behavior in value fields. Note that if $\alpha\mathfrak{o}_E \in P^{\mathfrak{m}}$ has order $r \bmod P_1^{\mathfrak{m}}$, then $\psi(\alpha\mathfrak{o}_E) = \zeta\alpha^\ell$ for some $\zeta \in \mu_r$. Since $\alpha^\ell \in L$, this means that in fact $\zeta \in \mu_L$.

Proposition 3.1. *For any nonzero ideal $\mathfrak{a} \subset \mathfrak{o}_E$, the L -ideal class of $\mathfrak{a}\mathfrak{o}_L$ has order dividing ℓ in $\text{Cl}(L)$.*

When $\ell = 1$, this means that L is a principalization (or capitulation) field for E .

Proof. We may assume \mathfrak{a} is coprime to \mathfrak{m} .

Consider $\mathfrak{a} \in J^{\mathfrak{m}}$. Say \mathfrak{a} has order r in $\text{Cl}(E)$, and write $\mathfrak{a}^r = \alpha\mathfrak{o}_E$, where $\alpha \in E^\times$. Then

$$(\psi(\mathfrak{a})\mathfrak{o}_L)^r = \psi(\mathfrak{a}^r)\mathfrak{o}_L = \alpha^\ell\mathfrak{o}_L = (\alpha\mathfrak{o}_L)^\ell = (\mathfrak{a}\mathfrak{o}_L)^{r\ell}.$$

By unique factorization of ideals, we have $(\mathfrak{a}\mathfrak{o}_L)^\ell = \psi(\mathfrak{a})\mathfrak{o}_L$, which is principal. \square

Corollary 3.2. *The exponent of $\text{Cl}(E)$ divides ℓd .*

Proof. For any ideal $\mathfrak{a} \subset \mathfrak{o}_E$, note that $\mathcal{N}_{L/E}(\mathfrak{a}\mathfrak{o}_L) = \mathfrak{a}^d$. Now suppose \mathfrak{a} is nonzero, and thus $\mathfrak{a}^\ell \mathfrak{o}_L = \beta \mathfrak{o}_L$ for some $\beta \in \mathfrak{o}_L$ by [Proposition 3.1](#). Hence $\mathfrak{a}^{\ell d} = \mathcal{N}_{L/E}(\mathfrak{a}^\ell \mathfrak{o}_L) = N(\beta) \mathfrak{o}_E$ is principal. \square

The above results generalize the case of $L = E$ in [\[Sch09\]](#). They allow us to reduce the problem of classifying degree d rationality fields of CM forms in $S_k(N)$ to considering those with CM by a field E with (class group) exponent dividing $(k-1)d$.

3.2. Restrictions on value fields. Now for fixed E and weight ℓ , we obtain necessary conditions on the possible value fields L .

Lemma 3.3. *Let $L_0 \subset L$ be the subfield generated by the values of ψ on principal ideals in $P^\mathfrak{m}$. Then $L_0 = E\mathbb{Q}(\zeta_r)$, where r is the order of η_ψ .*

Proof. First note that $\psi(P_1^\mathfrak{m})$ generates E . For any principal ideal in $P^\mathfrak{m}$, we may scale it by an element of $P_1^\mathfrak{m}$ so that it is represented by some $\alpha \in \mathfrak{o}_E$. Then $\eta_\psi(\alpha) = \psi(\alpha \mathfrak{o}_E)/\alpha^\ell \in L_0$. Conversely, all values of η_ψ must also lie in L_0 , and all values of ψ on $P^\mathfrak{m}$ must lie in $\eta_\psi((\mathfrak{o}_E/\mathfrak{m})^\times)E$. \square

For $\alpha \in E^\times$, let $\alpha^{1/n}$ denote some n -th root of α in the algebraic closure of E .

Proposition 3.4. *Let $\mathfrak{t}_1, \dots, \mathfrak{t}_g$ be representatives for a set of generators for $\text{Cl}(E)$. Let n_i be the order of \mathfrak{t}_i in $\text{Cl}(E)$ and write $\mathfrak{t}_i^{n_i} = \theta_i \mathfrak{o}_E$. Let r be the order of η_ψ . Then*

$$L = L_\psi = E(\zeta_r, (u_1 \theta_1^\ell)^{1/n_1}, \dots, (u_g \theta_g^\ell)^{1/n_g})$$

for some $u_1, \dots, u_g \in \mu_r$ (and some choices of n_i -th roots of $u_i \theta_i^\ell$). In particular, $L \subset \tilde{E}_{r,\ell} := E(\zeta_r, \zeta_n, \theta_1^{\ell/n}, \dots, \theta_g^{\ell/n})$ where n is the exponent of $\text{Cl}(E)$.

Note that $\tilde{E}_{r,\ell}$ really only depends on E , r and ℓ , and not on the choices of \mathfrak{t}_i or θ_i or n_i -th roots.

Proof. For $1 \leq i \leq g$, select $\mathfrak{a}_i = \gamma_i \mathfrak{t}_i \in J^\mathfrak{m}$ for some $\gamma_i \in E^\times$. Then $\mathfrak{a}_i^{n_i} = \alpha_i \mathfrak{o}_E$, where $\alpha_i = \gamma_i^{n_i} \theta_i$. From [Lemma 3.3](#), the values of ψ on principal ideals generates $L_0 = E(\zeta_r)$. Hence $L_\psi = L_0(\psi(\mathfrak{a}_1), \dots, \psi(\mathfrak{a}_g))$. Note that $\psi(\mathfrak{a}_i)$ is an n_i -th root of $\eta_\psi(\alpha_i) \alpha_i^\ell = \eta_\psi(\alpha_i) \theta_i^\ell \gamma_i^{n_i \ell}$. Hence $L_0(\psi(\mathfrak{a}_i)) = L_0((\eta_\psi(\alpha_i) \theta_i^\ell)^{1/n_i})$. \square

Remark 3.5. Fix E, ℓ and r as above. From the proof we see that L_ψ only depends on η_ψ and the choice of certain n_i -th roots of unity. If $\text{Cl}(E)$ is cyclic, there is only one n_i and any choice of an n_i -th root will give the same L_ψ up to isomorphism. If $n \mid r$, then the choice of root of unity does not affect the value field. Hence if either $n \mid r$ or $\text{Cl}(E)$ is cyclic, then η_ψ determines L_ψ up to isomorphism.

Once we bound the degree of the value field L_ψ , there are finitely many possibilities for the order of η_ψ (without restriction on ℓ or \mathfrak{m}) by [Lemma 3.3](#). Noting that the field $\tilde{E}_{r,\ell}$ in the proposition does not depend upon the conductor, and only depends on $\ell \bmod n$, we deduce the following.

Corollary 3.6. *Given E , $d \geq 1$, there are finitely many extensions L/E of degree d which are value fields of Grössencharacters of E (for arbitrary weights and conductors).*

This, together with [Corollary 3.2](#) and the finiteness results about imaginary quadratic fields with given exponent mentioned in the Introduction, proves [Theorem 1](#).

4. MINIMAL CONDUCTORS

In order to classify CM newforms with degree d rationality field (assuming a classification of E with given exponent), we would like to be able to do the following. Given E , d and ℓ , classify: (i) degree $2d$ value fields L of weight ℓ Grössencharacters ψ such that $\eta_\psi^\mathbb{Z} = \chi_E$, and (ii) all Grössencharacters ψ with a given value field L such that $\eta_\psi^\mathbb{Z} = \chi_E$. While (i) is our goal in this paper, we will say some things about (ii). In fact a large part of (ii) is determining what conductors \mathfrak{m} are possible, and this is useful for (i). Problem (ii) was taken up in [Sch09] when $L = E$, which corresponds to $K = \mathbb{Q}$. In this section we establish some results about what conductors need to be considered.

Let ψ be a Grössencharacter of E with weight ℓ and conductor \mathfrak{m} .

We say that ψ (resp. η_ψ) is primitive of conductor \mathfrak{m} if there is no integral ideal \mathfrak{m}' properly dividing \mathfrak{m} such that ψ extends to a Grössencharacter of conductor \mathfrak{m}' (resp. η_ψ factors through $(\mathfrak{o}_E/\mathfrak{m}')^\times$). For our classification problems, it clearly suffices to restrict to primitive ψ . The following says that ψ is primitive if and only if η_ψ is primitive.

Lemma 4.1. *Suppose \mathfrak{m}' is a nonzero integral ideal of E such that $\mathfrak{m}' \mid \mathfrak{m}$ and that η_ψ factors through $(\mathfrak{o}_E/\mathfrak{m}')^\times$. Then ψ is the restriction to $J^\mathfrak{m}$ of a Grössencharacter ψ' with conductor \mathfrak{m}' such that $\eta_{\psi'} = \eta_\psi$ and $L_{\psi'} = L_\psi$.*

Proof. By induction, it suffices to prove this in the case that $\mathfrak{m} = \mathfrak{p}\mathfrak{m}'$. If $\mathfrak{p} \mid \mathfrak{m}'$, then $J^{\mathfrak{m}'} = J^\mathfrak{m}$ and one only needs to check that $\psi(\alpha) = \alpha^\ell$ when $\alpha \equiv 1 \pmod{\mathfrak{m}'}$. But this is immediate from η being trivial on such α .

Assume $\mathfrak{m} = \mathfrak{p}\mathfrak{m}'$ where $\mathfrak{p} \nmid \mathfrak{m}'$. To extend ψ to a character ψ' of $J^{\mathfrak{m}'}$, we only need to specify $\psi'(\mathfrak{p})$. Write $\mathfrak{p} = \alpha\mathfrak{q}$, where $\alpha \in E^\times$ and $\mathfrak{q} \in J^\mathfrak{m}$. Then α is coprime to \mathfrak{m}' , and we can define $\psi'(\mathfrak{p}) = \eta(\alpha)\alpha^\ell\psi(\mathfrak{q})$.

To see this is independent of the choice of α and \mathfrak{q} , consider another representation $\mathfrak{p} = \alpha'\mathfrak{q}'$ where $\mathfrak{q}' \in J^\mathfrak{m}$. Then $\mathfrak{q}(\mathfrak{q}')^{-1} = \alpha'\alpha^{-1}\mathfrak{o}_E \in J^\mathfrak{m}$. From this one deduces $\eta(\alpha')(\alpha')^\ell\psi(\mathfrak{q}') = \eta(\alpha)\alpha^\ell\psi(\mathfrak{q}) = \psi'(\mathfrak{p})$.

Again, η being trivial on $E_1^{\mathfrak{m}'}$ means that ψ' has conductor \mathfrak{m}' . The assertions on $\eta_{\psi'}$ and $L_{\psi'}$ are also clear. \square

Given a Dirichlet character χ of conductor Q , we define the twist of ψ to be the primitive weight ℓ Grössencharacter of E such that $(\psi \otimes \chi)(\mathfrak{a}) = \chi(\mathcal{N}(\mathfrak{a}))\psi(\mathfrak{a})$ for $\mathfrak{a} \in J^{\mathfrak{m}Q}$. A priori its conductor is just a divisor of $\mathfrak{m}Q$, however $\psi \otimes \chi$ may have conductor which is a proper divisor of \mathfrak{m} .

Suppose χ is a quadratic character. Then the quadratic twist $\psi \otimes \chi$ has the same value field as ψ . (More generally, if χ has image contained in μ_L , then $L_{\psi \otimes \chi} \subset L_\psi$, but they are not in general equal.) Further $\eta_{\psi \otimes \chi}^\mathbb{Z} = \eta_\psi^\mathbb{Z}$. Consequently in our study of value fields, we may reduce to considering primitive Grössencharacters which have minimal conductor among the family of all quadratic twists $\psi \otimes \chi$.

Remark 4.2. By only working with newforms of trivial nebentypus or newforms with a fixed rationality field K , we cannot always restrict to considering minimal CM forms or Grössencharacters. For instance, there is a newform $f \in S_2(15^2)$ (labelled 225.2.a.f in the LMFDB) with rationality field $\mathbb{Q}(\sqrt{5})$ and CM by $\mathbb{Q}(\sqrt{-15})$, but it is not minimal: it is an order 4 twist of a level 45 form with character $\left(\frac{5}{\cdot}\right)$ that has rationality field $\mathbb{Q}(\sqrt{-5})$.

Now let us restrict to the setting that $\eta_\psi^\mathbb{Z} = \chi_E$. We have the following minimal divisibility condition on the conductor. For this and subsequent results, we will need some facts about the structure of modular multiplicative groups $(\mathfrak{o}_E/\mathfrak{p}^e)^\times$. These facts are well known for p odd, and are recalled in [Appendix A](#) below for $p = 2$.

Lemma 4.3. *If $\eta_\psi^\mathbb{Z} = \chi_E$ then $\mathfrak{d}_E \mid \mathfrak{m}$, where \mathfrak{d}_E is defined by*

$$\mathfrak{d}_E := \begin{cases} \sqrt{\Delta_E} \mathfrak{o}_E & \text{if } 2 \nmid \Delta_E \\ \mathfrak{p}_2 \sqrt{\Delta_E} \mathfrak{o}_E & \text{if } 4 \parallel \Delta_E \\ 2\sqrt{\Delta_E} \mathfrak{o}_E & \text{if } 8 \parallel \Delta_E, \end{cases}$$

and \mathfrak{p}_2 is the prime above 2 in E when $2 \mid \Delta_E$.

Proof. Since χ_E is a primitive Dirichlet character of conductor $|\Delta_E|$, we need that the natural map $(\mathbb{Z}/\Delta_E\mathbb{Z})^\times \rightarrow (\mathfrak{o}_E/\mathfrak{m})^\times$ is injective. Write $\Delta_E = -2^e q_1 \dots q_t$, where the q_i 's are odd primes and $e \in \{0, 2, 3\}$. Then $(\mathbb{Z}/\Delta_E\mathbb{Z})^\times \simeq (\mathbb{Z}/2^e\mathbb{Z})^\times \times \prod (\mathbb{Z}/q_i\mathbb{Z})^\times$. Using the analogous Chinese Remainder Theorem decomposition for $(\mathfrak{o}_E/\mathfrak{m})^\times$, we see that $(\mathbb{Z}/p\mathbb{Z})^\times$ injects into $(\mathfrak{o}_E/\mathfrak{p}^{v_p(\mathfrak{m})})^\times$ for any $p \mid \Delta_E$.

Suppose now that Δ_E is even. Since $(\mathbb{Z}/4\mathbb{Z})$ (resp. $(\mathbb{Z}/8\mathbb{Z})^\times$) embeds in $(\mathfrak{o}_E/\mathfrak{p}_2^j)^\times$ if and only if $j \geq 3$ (resp. $j \geq 5$), we must have that $\mathfrak{p}_2^3 \mid \mathfrak{m}$ when $4 \parallel \Delta_E$ and $\mathfrak{p}_2^5 \mid \mathfrak{m}$ when $8 \parallel \Delta_E$. Thus in all cases $\mathfrak{d}_E \mid \mathfrak{m}$. \square

Consequently, if $f \in S_k(N)$ is a newform with CM by E , we must have $N_E \mid N$, where

$$N_E := \Delta_E \cdot \mathcal{N}(\mathfrak{d}_E) = \begin{cases} \Delta_E^2 & \text{if } 2 \nmid \Delta_E \\ 2\Delta_E^2 & \text{if } 4 \parallel \Delta_E \\ 4\Delta_E^2 & \text{if } 8 \parallel \Delta_E. \end{cases}$$

In particular, if we want to determine the existence of a Grössencharacter ψ of E with value field $L = EK$ (resp. CM form f with rationality field K), the minimal conductor (resp. level) in which it can occur is \mathfrak{d}_E (resp. N_E). Under certain conditions, we can reduce to conductor \mathfrak{d}_E (resp. level N_E):

Proposition 4.4. *Suppose ψ is primitive with $\eta_\psi^2 = 1$ and $\eta_\psi^\mathbb{Z} = \chi_E$. Then there is a quadratic twist $\psi \otimes \chi$ which has conductor \mathfrak{d}_E . Moreover, either Δ_E is odd or $\Delta_E \equiv 0 \pmod{8}$.*

This is essentially proven in [\[Sch09\]](#), though it is formulated in the setting of $L = E$.

Proof. Say $\mathfrak{m} = \mathfrak{p}^e \mathfrak{m}'$, where \mathfrak{m}' is coprime to \mathfrak{p} . Let p be the prime below \mathfrak{p} . Decompose $(\mathfrak{o}_E/\mathfrak{m})^\times = (\mathfrak{o}_E/\mathfrak{p}^e)^\times \times (\mathfrak{o}_E/\mathfrak{m}')^\times$. Write $\eta = \eta_\psi$, and let $\eta_\mathfrak{p}$ be the character of $(\mathfrak{o}_E/\mathfrak{p}^e)^\times$ obtained by restricting η to this factor.

We may assume ψ , and thus η , is primitive. Since $(\mathfrak{o}_E/\mathfrak{p}^e)^\times$ is isomorphic to $(\mathfrak{o}_E/\mathfrak{p}_E)^\times$ times a group of order q^{e-1} , where $q = \mathcal{N}(\mathfrak{p})$, $\eta_\mathfrak{p}$ must factor through $(\mathfrak{o}_E/\mathfrak{p}^e)^\times$ if q is odd. Hence we may assume $e \leq 1$ when p is odd. If $e = 1$, primitivity means $\eta_\mathfrak{p}$ is a (nontrivial) quadratic character.

Next suppose that p is an odd prime such that $p \mid \mathcal{N}(\mathfrak{m})$ and $p \nmid \Delta_E$. We show we can twist away the p -part of the conductor.

Consider the natural map $(\mathbb{Z}/p\mathbb{Z})^\times \rightarrow (\mathfrak{o}_E/p\mathfrak{o}_E)^\times$, and let η_p be the restriction of η to $(\mathfrak{o}_E/p\mathfrak{o}_E)^\times$. Then η_p must be trivial on the image of this map.

If $p\mathfrak{o}_E = \mathfrak{p}\bar{\mathfrak{p}}$ is split in E , then identifying $(\mathfrak{o}_E/p\mathfrak{o}_E)^\times = (\mathfrak{o}_E/\mathfrak{p})^\times \times (\mathfrak{o}_E/\bar{\mathfrak{p}})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times$, we see $\eta_p(a, a) = 1$ for all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$. Hence $\eta_p(a, b) = \chi_p(a)\chi_p(b)$, where χ_p denotes the unique quadratic Dirichlet character of conductor p . (We remark this implies that if $\mathfrak{p} \mid \mathfrak{m}$, so does $\bar{\mathfrak{p}}$.) Since we also have $\eta_p = \chi_p \circ \mathcal{N}$ as characters of $(\mathfrak{o}_E/p\mathfrak{o}_E)^\times$, the twist $\psi \otimes \chi_p$ must have conductor $p^{-1}\mathfrak{m}$.

If p is inert in E , then $(\mathfrak{o}_E/\mathfrak{p})^\times \simeq C_{p^2-1}$, and η_p factors through C_{p^2-1}/C_{p-1} . Again, this must match with $\chi_p \circ \mathcal{N}$, so that $\psi \otimes \chi_p$ has conductor $\mathfrak{p}^{-1}\mathfrak{m}$.

Finally we treat the case $p = 2$.

First suppose that $2 \mid \Delta_E$. By [Lemma 4.3](#), we have that $e \geq 3$ if $4 \parallel \Delta_E$ and $e \geq 5$ if $8 \mid \Delta_E$. If $4 \parallel \Delta_E$, -1 is the square of an element of order 4 in $(\mathfrak{o}_E/\mathfrak{p}^e)^\times$ which forces η_ψ to have order a multiple of 4. Hence $4 \parallel \Delta_E$ is impossible. If $8 \mid \Delta_E$, then only $e = 5$ is possible for η_p to be primitive and quadratic.

Now assume that $p = 2 \nmid \Delta_E$. Then η_p can only be quadratic and primitive if $e \leq 3$.

If $2 = \mathfrak{p}\bar{\mathfrak{p}}$ splits in E , then as in the p odd case we may identify $\eta_p = \eta_{\bar{\mathfrak{p}}}$ as the same (possibly imprimitive) character of $(\mathbb{Z}/8\mathbb{Z})^\times \simeq \mathbb{C}_2^2$. Viewing $\chi = \eta_p$ as a Dirichlet character of conductor dividing 8, we see that $\psi \otimes \chi$ has conductor coprime to 2.

If 2 is inert in E , then η_p factors through $(\mathfrak{o}_E/8\mathfrak{o}_E)^\times$ modulo the image of $(\mathbb{Z}/8\mathbb{Z})^\times$. This quotient is isomorphic to $C_3 \times C_2^2$, so η_p factors through C_2^2 . Since E_2/\mathbb{Q}_2 is unramified, the norm induces a surjective map $(\mathfrak{o}_E/8\mathfrak{o}_E)^\times \rightarrow (\mathbb{Z}/8\mathbb{Z})^\times$, and hence η_p must be of the form $\chi \circ \mathcal{N}$ for one of the quadratic Dirichlet characters χ of conductor dividing 8. Hence $\psi \otimes \chi$ for this χ has conductor coprime to 2. \square

Corollary 4.5. *Let $\ell \geq 1$ be odd and $\mathcal{A}_\ell^2(E)$ be the set weight ℓ Grössencharacter ψ of E with quadratic modulus character η_ψ such that $\eta_\psi^\mathbb{Z} = \chi_E$. If $\text{Cl}(E)$ is cyclic or has exponent 2, then the set $\{L_\psi : \psi \in \mathcal{A}_\ell^2(E)\}$ of value fields up to isomorphism has cardinality (i) 1 if Δ_E is odd, (ii) 0 if $4 \parallel \Delta_E$, and (iii) 1 or 2 if $8 \parallel \Delta_E$.*

Proof. There are no such ψ when in case (ii), so assume we are in case (i) or (iii). By [Proposition 4.4](#), we may assume ψ has modulus \mathfrak{d}_E .

We have $(\mathfrak{o}_E/\mathfrak{d}_E)^\times \simeq (\mathbb{Z}/\Delta_E\mathbb{Z})^\times$ when Δ_E is odd and $(\mathfrak{o}_E/\mathfrak{d}_E)^\times \simeq (\mathbb{Z}/\Delta_E\mathbb{Z})^\times \times C_4$ when $8 \mid \Delta_E$. Hence there is 1 possibility for η_ψ when Δ_E is odd and 2 possibilities for a quadratic η_ψ when $8 \mid \Delta_E$ (the generator of the C_4 factor must map to ± 1). Now apply [Remark 3.5](#). \square

5. VALUE FIELDS FOR CLASS NUMBER 1

Suppose $h_E = 1$, i.e., $-\Delta_E \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$. Then for fixed \mathfrak{m} and ℓ , the map $\psi \mapsto \eta_\psi$ defines a bijection of Grössencharacters ψ with conductor \mathfrak{m} and weight ℓ and characters $\eta : (\mathfrak{o}_E/\mathfrak{m})^\times / (\mathfrak{o}_E^\times \cap E_1^\mathfrak{m}) \rightarrow \mathbb{C}^\times$. Further, by [Lemma 3.3](#), $L_\psi = \mathbb{Q}(\zeta_r)E$ where r is the order of η_ψ . Recall from [Section 2.2](#) that $(\mathfrak{o}_E^\times \cap E_1^\mathfrak{m})$ is trivial except in a few small cases.

We explicitly exhibit all fields of the form $L = \mathbb{Q}(\zeta_r)E$ with $d = [L : E] \leq 3$ as value fields of Grössencharacters ψ such that $\eta_\psi^\mathbb{Z} = \chi_E$. It suffices to produce a character η whose values generate L and whose restriction to \mathbb{Z} is χ_E . Then for any odd $\ell \geq 1$, η determines a weight ℓ Grössencharacter such that $\eta_\psi = \eta$. We may assume r is even, and construct η as follows.

From [Lemma 4.3](#), we need to work with conductors \mathfrak{m} which are multiples of \mathfrak{d}_E . We will usually choose conductors of the form $\mathfrak{m} = \mathfrak{p}^e \mathfrak{d}_E$ where \mathfrak{p} is coprime to \mathfrak{d}_E . Then it

suffices to choose \mathfrak{p}^e such that $(\mathfrak{o}_E/\mathfrak{p}^e)^\times/(\mathbb{Z}/p^e\mathbb{Z})^\times$ (meaning the quotient by the image of the natural map²) has a character λ of order r , or $\frac{r}{2}$ if $r \equiv 2 \pmod{4}$. Hence prescribing η to be $\chi_E \bmod \Delta_E$ and $\lambda \bmod \mathfrak{p}^e$ makes η a character of order r and conductor \mathfrak{m} whose restriction to \mathbb{Z} is χ_E .

5.1. $d = 1$. First we explain how to construct η with values in $L = E$. This typically means we can take η to be quadratic, and usually we may take $\mathfrak{m} = \mathfrak{d}_E$.

If $\Delta_E < -4$ is odd, we may take $\mathfrak{m} = \mathfrak{d}_E = \sqrt{\Delta_E}\mathfrak{o}_E$ and then $\mathfrak{o}_E^\times \cap E_1^\mathfrak{m} = \{1\}$. Since $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq (\mathbb{Z}/\Delta_E\mathbb{Z})^\times$, let $\eta = \chi_E$.

If $E = \mathbb{Q}(i)$, take $\mathfrak{m} = \mathfrak{d}_E = 2(1+i)\mathfrak{o}_E$. Then $\mathfrak{o}_E^\times \cap E_1^\mathfrak{m} = \{1\}$, and $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq \{\pm 1, \pm i\}$. We can take η to be either order 4 character of $(\mathfrak{o}_E/\mathfrak{m})^\times$, and it restricts to the nontrivial quadratic character on $(\mathbb{Z}/4\mathbb{Z})^\times \simeq \{\pm 1\}$.

If $E = \mathbb{Q}(\zeta_3)$, take $\mathfrak{m} = \mathfrak{d}_E^2 = 3\mathfrak{o}_E$. Here we do not take $\mathfrak{m} = \mathfrak{d}_E$ because $\mathfrak{o}_E^\times \cap E_1^{\mathfrak{d}_E} \neq \{1\}$. However $\mathfrak{o}_E^\times \cap E_1^\mathfrak{m} = \{1\}$, and $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq \langle \zeta_6 \rangle$. Take η to be any even order character of $(\mathfrak{o}_E/\mathfrak{m})^\times$, which restricts to the nontrivial quadratic character on $(\mathbb{Z}/(\mathfrak{m} \cap \mathbb{Z}))^\times \simeq \{\pm 1\}$.

If $E = \mathbb{Q}(\sqrt{-2})$, take $\mathfrak{m} = \mathfrak{d}_E = 4\sqrt{-2}\mathfrak{o}_E$. Then $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq (\mathbb{Z}/8\mathbb{Z})^\times \times C_4$, and we may take η to be either quadratic character restricting to χ_E on $(\mathbb{Z}/8\mathbb{Z})^\times$.

In all of these cases, for any odd $\ell \geq 1$, η determines a Grössencharacter ψ of E with conductor \mathfrak{m} and weight ℓ such that $\eta_\psi = \chi_E$ and $L_\psi = E$. In particular, one gets a CM form $f_\psi \in S_{\ell+1}(N)$ with rationality field $K = \mathbb{Q}$, where $N = N_E$ or 27 according to whether $\Delta_E \neq -3$ or $\Delta_E = -3$.

5.2. $d = 2$. Now we construct η whose values generate a quadratic extension $L = \mathbb{Q}(\zeta_r)E$ of E .

Assume first that $\Delta_E < -4$. Then $\mu_E = \{\pm 1\}$, and the only quadratic extensions of the form $L = \mathbb{Q}(\zeta_r)E$ occur when $r = 4, 6$.

We first treat $r = 4$, i.e., $L = E(i)$.

Except in the case of $\Delta_E = -8$ and $\Delta_E = -11$, the prime 3 is inert in E , and thus we take $\mathfrak{m} = 3\mathfrak{d}_E$. Then $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq (\mathfrak{o}_E/3\mathfrak{o}_E)^\times \times (\mathfrak{o}_E/\mathfrak{d}_E)^\times$. We define η to be one of the order 4 characters on the $(\mathfrak{o}_E/3\mathfrak{o}_E)^\times/(\mathbb{Z}/3\mathbb{Z})^\times \simeq C_4$ component and χ_E on $(\mathfrak{o}_E/\mathfrak{d}_E)^\times \simeq (\mathbb{Z}/\Delta_E\mathbb{Z})^\times$ component.

When $\Delta_E = -8$, we can take $\mathfrak{m} = \mathfrak{d}_E$ since $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq (\mathbb{Z}/8\mathbb{Z})^\times \times C_4$, and let η be order 4 on the C_4 component.

When $\Delta_E = -11$ (and also when $\Delta_E = -19$), 5 is split in E and we can take $\mathfrak{m} = 5\mathfrak{d}_E$. Then we take η to be of the form $\lambda(a, b) = \lambda(a)\lambda(b)$ on the factor $(\mathfrak{o}_E/5\mathfrak{o}_E)^\times \simeq (\mathbb{Z}/5\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times$, where λ is an order 4 character.

Hence for each $\Delta_E < -4$ we get a Grössencharacter ψ of conductor \mathfrak{m} and any odd weight ℓ with value field $L = E(i)$. Then the associated CM form f_ψ has rationality field $K = \mathbb{Q}(\sqrt{|\Delta_E|})$ and level $9N_E$ or N_E , or $25N_E$, according to the 3 cases above.

Next we treat $r = 6$, i.e., $L = E(\zeta_3)$.

Except for $\Delta_E = -7, -8$, one sees that 2 is inert in E . Thus there exists an order 6 character $\eta \bmod \mathfrak{m} = 2\mathfrak{d}_E$ which is compatible with χ_E .

²For simplicity, from now on if $n \in \mathbb{Z}$ and \mathfrak{n} is an ideal in \mathfrak{o}_E such that the natural map $(\mathbb{Z}/n\mathbb{Z})^\times$ to $(\mathfrak{o}_E/\mathfrak{n})^\times$ is an injection, we identify $(\mathbb{Z}/n\mathbb{Z})^\times$ with its image in $(\mathfrak{o}_E/\mathfrak{n})^\times$.

Except for $\Delta_E = -11, -19$, the prime 5 is inert in E . Hence there exists an order 6 character $\eta \bmod \mathfrak{m} = 5\mathfrak{d}_E$ which is compatible with χ_E .

In these situations we get a Grössencharacter ψ of conductor \mathfrak{m} and any odd weight ℓ with value field $L = E(\zeta_3)$. Then the associated CM form f_ψ has rationality field $K = \mathbb{Q}(\sqrt{3|\Delta_E|})$ and level $4N_E$ or $25N_E$.

Finally let us consider $\Delta_E = -4, -3$.

If $E = \mathbb{Q}(i)$, then one can have $L = \mathbb{Q}(\zeta_8)$ or $L = \mathbb{Q}(\zeta_{12})$. Since 7 and 11 are inert in E , we can achieve each of these value fields with conductors $\mathfrak{m} = 7\mathfrak{d}_E$ and $\mathfrak{m} = 11\mathfrak{d}_E$, respectively. This produces CM forms in even weights $k \geq 2$ of levels $2^5 \cdot 7^2$ and $2^5 \cdot 11^2$ with rationality fields $K = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(\sqrt{3})$.

If $E = \mathbb{Q}(\zeta_3)$, then one can have $L = \mathbb{Q}(\zeta_{12})$. Again, 11 is inert in E , so we obtain this value field in conductor $\mathfrak{m} = 11\mathfrak{d}_E$. This yields a CM form in even weights $k \geq 2$ of level $3^2 \cdot 11^2$ with rationality field $\mathbb{Q}(\sqrt{3})$.

5.3. $d = 3$. Now we construct η whose values generate a cubic extension $L = \mathbb{Q}(\zeta_r)E$ of E . There are no degree 3 cyclotomic extensions of \mathbb{Q} , so the only possibility is $E \subset \mathbb{Q}(\zeta_r)$ and $\mathbb{Q}(\zeta_r)$ has degree 6. Thus either $E = \mathbb{Q}(\zeta_3)$ and $L = \mathbb{Q}(\zeta_9)$ or $E = \mathbb{Q}(\sqrt{-7})$ and $L = \mathbb{Q}(\zeta_7)$.

Suppose $E = \mathbb{Q}(\zeta_3)$ and $L = \mathbb{Q}(\zeta_9)$. Take $\mathfrak{m} = \mathfrak{d}_E^6 = 27\mathfrak{o}_E$. Then $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq \langle \zeta_{18} \rangle \simeq (\mathbb{Z}/3\mathbb{Z})^\times \times C_9$. Define η to be the quadratic character on $(\mathbb{Z}/3\mathbb{Z})^\times$ and order 9 on C_9 . This gives a Grössencharacter ψ for any odd weight ℓ with $\eta_\psi^\mathbb{Z} = \chi_E$ and $L_\psi = L$. Then $f_\psi \in S_{\ell+1}(3^7)$ and has rationality field $K = \mathbb{Q}(\zeta_9)^+$.

Suppose $E = \mathbb{Q}(\sqrt{-7})$ and $L = \mathbb{Q}(\zeta_7)$. We can take $\mathfrak{m} = \mathfrak{d}_E^2 = 7\mathfrak{o}_E$. Then $(\mathfrak{o}_E/\mathfrak{m})^\times \simeq (\mathbb{Z}/7\mathbb{Z})^\times \times C_7$. Define η to be the quadratic character on $(\mathbb{Z}/7\mathbb{Z})^\times$ and order 7 on C_7 . This gives a Grössencharacter ψ for any odd weight ℓ with $\eta_\psi^\mathbb{Z} = \chi_E$ and $L_\psi = L$. One has $f_\psi \in S_{\ell+1}(7^3)$ with rationality field $K = \mathbb{Q}(\zeta_7)^+$.

6. VALUE FIELDS WITH QUADRATIC MODULAR CHARACTERS

Here we will determine value fields L of Grössencharacters ψ of E such that η_ψ is quadratic and $\eta_\psi^\mathbb{Z} = \chi_E$, when E has exponent 2 or 3. By [Proposition 4.4](#), we may assume $\Delta_E \not\equiv 4 \pmod{8}$ and the conductor of ψ is $\mathfrak{m} = \mathfrak{d}_E$. Hence the associated CM form f_ψ will have level N_E .

Suppose $d = [L : E]$ is prime and ψ has odd weight $\ell \geq 1$. Let $\mathfrak{t}_1, \dots, \mathfrak{t}_g$ be representatives for a set of generators for $\text{Cl}(E)$ which are coprime to \mathfrak{d}_E . Let n_i be the order of \mathfrak{t}_i in $\text{Cl}(E)$ and write $\mathfrak{t}_i^{n_i} = \theta_i \mathfrak{o}_E$. By [Proposition 3.4](#), for each $1 \leq i \leq g$ we must have $L_\psi = E((\pm\theta_i)^{\ell/n_i})$ for some choice of \pm (which may depend upon i).

Assuming ℓ is coprime to h_E we have the following necessary condition:

- (Q1) if $\text{Cl}(E)$ is not cyclic, there exists a collections of signs $\varepsilon_1, \dots, \varepsilon_g \in \{\pm 1\}$ such that $E((\varepsilon_i \theta_i^\ell)^{1/n_i}) = E((\varepsilon_j \theta_j^\ell)^{1/n_j})$ for all $1 \leq i < j \leq g$.

6.1. Exponent 2 fields. There are 56 known imaginary quadratic fields E with (class group) exponent 2, with the discriminant of largest absolute value being -5460 , and this list is complete under ERH [[Wei73](#)]. Of these, 18 have class group C_2 , 24 have class group C_2^2 , 13 have class group C_2^3 , and 1 has class number C_2^4 .

For those with class number $h_E > 2$, we check that (Q1) never holds for ℓ odd. (Note that we may assume $\ell = 1$ for this.)

Δ_E	-15	-35	-51	-91	-115	-123	-187	-235	-267	-403	-427
Δ_K	5	5	17	13	5	41	17	5	89	13	61

TABLE 6.1. Value fields $L = EK$ when Δ_E is odd

Δ_E	-24	-40	-88	-232
Δ_K	8	5	8	29

TABLE 6.2. Value fields $L = EK$ when Δ_E is even

Among those with class number 2, there are 15 with $\Delta_E \not\equiv 4 \pmod{8}$, namely those with

$$-\Delta_E \in \{15, 24, 35, 40, 51, 88, 91, 115, 123, 187, 232, 235, 267, 403, 427\}.$$

Assume E is one of these 15 fields. Since $g = 1$, write $\theta = \theta_1$. Then $L = E(\sqrt{\eta_\psi(\theta)\bar{\theta}})$, so to determine the value fields for each E we just compute a representative θ and $\eta(\theta)$ for each character η of $(\mathfrak{o}_E/\mathfrak{d}_E)^\times$ which restrict to χ_E on $(\mathbb{Z}/\Delta_E\mathbb{Z})^\times$. (Recall that [Corollary 4.5](#), there is a unique value field L when Δ_E is odd, and at most two value fields when $8 \mid \Delta_E$; we will see that there is a unique value field in all cases.)

When Δ_E is odd, $(\mathbb{Z}/\Delta_E\mathbb{Z})^\times$ surjects onto $(\mathfrak{o}_E/\mathfrak{d}_E)^\times$ and thus $\eta_\psi(\theta)$ is determined by χ_E . We compute the value fields $L = L_\psi$ for these Δ_E and record them in [Table 6.1](#). Since $L = EK$ is specified by E and its real quadratic field K , we just record Δ_K in this table.

Now suppose $\Delta_E \equiv 0 \pmod{8}$. Write $\Delta_E = -8q$, where $q \in \{3, 5, 11, 29\}$, and let \mathfrak{q} be the prime above q . Now the image of θ in $(\mathfrak{o}_E/\mathfrak{d}_E)^\times$ may not be in the image of $(\mathbb{Z}/\Delta_E\mathbb{Z})^\times$. Indeed, computation shows that in each case we can choose θ such that $\theta \equiv 2 \pmod{\mathfrak{q}}$ and $\theta \equiv -q\varepsilon^2 \pmod{\mathfrak{p}_2^5}$, where $\varepsilon = 1 + \sqrt{-2q}$ has order 4 mod \mathfrak{p}_2^5 . Since η is quadratic, one has that $\eta_2(\theta) = \eta_2(-q)$ where η_2 is η composed with the projection to $(\mathfrak{o}_E/\mathfrak{p}_2^5)^\times$. Thus $\eta(\theta) = \chi_E(t)$, where $t \in (\mathbb{Z}/\Delta_E\mathbb{Z})^\times$ such that $t \equiv 2 \pmod{q}$ and $t \equiv -q \pmod{8}$. (Hence the two possible η 's must will yield the same value field L .)

Explicitly for $q = 3, 5, 11, 29$, we can respectively take $t = 5, 27, 13, 147$ and find that $\eta(\theta) = -(\frac{-1}{q})$. We tabulate the value fields $L = EK$ via discriminants in [Table 6.2](#).

6.2. Exponent 3 fields. There are 17 known imaginary quadratic fields E with exponent 3, and there are no others if ERH holds [\[EKN20\]](#). All of these fields have negative prime discriminant, so Δ_E is odd. One of them, $\mathbb{Q}(\sqrt{-4027})$, has class group C_3^2 , and one checks this field does not satisfy [\(Q1\)](#) (assuming $3 \nmid \ell$, so one only needs to check $\ell = 1, 2$).

All the others have class group C_3 , and their discriminants satisfy

$$-\Delta_E = \{23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907\}.$$

For each these class number 3 fields, there is a unique value field L that occurs by [Corollary 4.5](#), which we can construct in a similar way to the Δ_E odd case of [Section 6.1](#). Namely, L is of the form $E((\eta(\theta)\theta^\ell)^{1/3})$, and it only depends on $\ell \pmod{3}$. In [Table 6.3](#) we indicate the value fields $L = EK$ for $\ell \equiv 1 \pmod{6}$, where K is a totally real cubic field with the listed discriminant. In all of these cases the discriminant uniquely identifies K up to isomorphism.

Δ_E	\parallel	-23	\parallel	-31	\parallel	-59	\parallel	-83	\parallel	-107	\parallel	-139	\parallel	-211	\parallel	-283
Δ_K	\parallel	621	\parallel	837	\parallel	1593	\parallel	2241	\parallel	321	\parallel	3753	\parallel	5697	\parallel	7641

Δ_E	\parallel	-307	\parallel	-331	\parallel	-379	\parallel	-499	\parallel	-547	\parallel	-643	\parallel	-883	\parallel	-907
Δ_K	\parallel	8289	\parallel	993	\parallel	10233	\parallel	13473	\parallel	14769	\parallel	1929	\parallel	23841	\parallel	24489

TABLE 6.3. Value fields $L = EK$ for $\ell \equiv 1 \pmod 6$

7. VALUE FIELDS WITH HIGHER ORDER MODULAR CHARACTERS

Lastly, we consider ψ such that η_ψ has order $r > 2$ and $\eta_\psi^{\mathbb{Z}} = \chi_E$. Combining the results below with those in the previous two sections will prove [Theorem 2](#).

Assume $h_E > 1$, so that $L_0 = E(\zeta_r) \neq E$. If $d = [L : E]$ is prime then $L = L_0$, and one needs the following:

(R1) For each $1 \leq i \leq g$, there exists $\zeta \in \mu_L$ such that $(\zeta \theta_i^\ell)^{1/n_i} \in E(\zeta_r)$.

7.1. Exponent 2 fields. Suppose E has exponent 2 and $L = L_0 = E(\zeta_r)$ has degree 2 over E . Then we can either take $r = 4$ or $r = 6$.

We check that (R1) holds only in the following situations:

- (i) $r = 4$, $-\Delta_E \in \{20, 24, 40, 52, 88, 148, 232\}$, and ζ is a primitive 4th root of unity; or
- (ii) $r = 6$: $-\Delta_E \in \{15, 24, 51, 123, 267\}$, and $\pm\zeta$ is a primitive 3rd root of unity.

All of these are class number 2 fields. Take a representative \mathfrak{t} for the nontrivial element of $\text{Cl}(E)$ which is coprime to Δ_E . Say $\mathfrak{t}^2 = \theta \mathfrak{o}_E$.

First suppose we are in case (i). Then 2 is ramified E , and let \mathfrak{p}_2 denote the prime above 2. To see whether a given E in (i) yields a ψ as above amounts to determining whether there exists a modulus \mathfrak{m} with character η which agrees with χ_E and satisfies $\eta(\theta) = \pm i$.

If $\Delta_E \equiv 4 \pmod 8$, then we compute in each case that $\theta^2 \equiv a \pmod{\mathfrak{d}_E}$, where $a \in (\mathbb{Z}/\Delta_E \mathbb{Z})^\times$ is an element on which χ_E is -1 . Specifically, for $D = -20, -52, -148$, we take θ to be an element of $E - \mathbb{Q}$ of norm 9, 49, 361 and find $a = -1, 23, -9$, respectively. Hence for these D , either character η of $(\mathfrak{o}_E/\mathfrak{d}_E)^\times$ which restricts to χ_E on $(\mathbb{Z}/\Delta_E \mathbb{Z})^\times$ sends θ to a primitive 4th root of unity, and thus it gives a Grössencharacter ψ for any odd weight ℓ such that $L_\psi = E(i)$. This gives CM forms $f \in S_{\ell+1}(2\Delta_E^2)$ with $K_f = \mathbb{Q}(\sqrt{|\Delta_E|})$.

Assume now $\Delta_E \equiv 0 \pmod 8$. We claim there is no modulus \mathfrak{m} with an order 4 character $\eta : (\mathfrak{o}_E/\mathfrak{m})^\times \rightarrow \mathbb{C}^\times$ such that η restricted to $(\mathbb{Z}/\mathcal{N}(\mathfrak{m}))^\times$ is χ_E and $\eta(\theta) = \pm i$. (Tacitly, when necessary we replace \mathfrak{t} and θ with representatives that are coprime to \mathfrak{m} .)

Suppose that such a \mathfrak{m} and η exist. We may assume η is primitive. Then there exists a prime $\mathfrak{p} \mid \mathfrak{m}$ such that $\eta_{\mathfrak{p}}(\theta) = \pm i$, where $\eta_{\mathfrak{p}}$ is η composed with the projection $(\mathfrak{o}_E/\mathfrak{m})^\times \rightarrow (\mathfrak{o}_E/\mathfrak{p}^e)^\times$ and $e = v_{\mathfrak{p}}(\mathfrak{m})$. Let \mathfrak{p} be such a prime and p be the prime below \mathfrak{p} . Necessarily, θ is primitive mod \mathfrak{p}^e , because if it were a nontrivial power of some $\beta \pmod{\mathfrak{p}^e}$, then $\eta_{\mathfrak{p}}(\beta)$ and hence η would have to have order strictly larger than 4.

First suppose that $\mathfrak{p} = \mathfrak{p}_2$ is the prime above 2. We compute that the minimal e such that $\theta \pmod{\mathfrak{p}_2^e}$ has order 4 is $e = 6$. However, in this case, $(\mathfrak{o}_E/\mathfrak{p}_2^6)^\times \simeq (\mathbb{Z}/8\mathbb{Z})^\times \times C_8$. Hence $\eta(\theta) \in \{\pm i\}$ would imply η has order ≥ 8 , and thus $\mathfrak{p} = \mathfrak{p}_2$ is impossible.

So assume p is odd. Then as in the proof of [Proposition 4.4](#), primitivity implies $e = 1$. Note that $\mathcal{N}(\theta) = \mathcal{N}(\mathfrak{t})^2$ is a square in $(\mathbb{Z}/p\mathbb{Z})^\times$. Since the norm map from $(\mathfrak{o}_E/\mathfrak{p})^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ sends the subgroup of squares to the subgroup of squares, this means that θ is a square mod \mathfrak{p} . Thus θ is not primitive mod \mathfrak{p} , and we have proved our claim by contradiction.

Now consider case (ii). Note that 3 is ramified in E . In each case, θ generates $(\mathfrak{o}_E/3\mathfrak{o}_E)^\times \simeq C_6$. Let \mathfrak{p}_3 be the prime in E above 3. Then for any odd ℓ , there is a Grössencharacter $\psi \bmod \mathfrak{m} := \mathfrak{p}_3\mathfrak{d}_E$ such that $L_\psi = E(\zeta_3)$ and $\eta_\psi^\mathbb{Z} = \chi_E$. The associated CM form $f_\psi \in S_{\ell+1}(3N_E)$ has rationality field $K = \mathbb{Q}(\sqrt{\frac{|\Delta_E|}{3}})$. For each of these E , this does not produce any value/rationality fields different from the case of quadratic modulus character in [Section 6.1](#), though we now get forms of level $3N_E$ instead of level N_E .

7.2. Exponent 3 fields. Suppose E has exponent 3 and $L = L_0 = E(\zeta_r)$ has degree 3 over E . Then L must be a cyclotomic field of degree 6, and E is the quadratic subfield. Namely $L = \mathbb{Q}(\zeta_7)$ and $E = \mathbb{Q}(\sqrt{-7})$ or $L = \mathbb{Q}(\zeta_9)$ and $E = \mathbb{Q}(\sqrt{-3})$. In both cases $h_E = 1$, violating our assumption. (We already constructed such Grössencharacters in [Section 5.3](#).)

APPENDIX A. STRUCTURE OF DYADIC MODULAR QUOTIENTS

Let E/\mathbb{Q} be a quadratic field, and \mathfrak{p} be a prime above 2. Here we recall some structure theory about $(\mathfrak{o}_E/\mathfrak{p}^n)^\times$. If 2 splits in E , then $(\mathfrak{o}_E/\mathfrak{p}^n)^\times \simeq (\mathbb{Z}/2^n\mathbb{Z})^\times \simeq C_2 \times C_2^{n-2}$ for $n \geq 2$. So assume 2 is either inert or ramified in E .

Consider a local quadratic extension k/\mathbb{Q}_2 with integer ring \mathfrak{o}_k and prime ideal \mathfrak{p}_k . Write $\mathfrak{u}_n = (1 + \mathfrak{p}^n) \cap \mathfrak{o}_k^\times$ for the n -th unit group of k . Put $f = f(k/\mathbb{Q}_2)$ and $e = e(k/\mathbb{Q}_2)$. It is well known that

$$G := (\mathfrak{o}_E/\mathfrak{p}^n)^\times = (\mathfrak{o}_k/\mathfrak{p}_k^n)^\times \simeq C_{2^f-1} \times \mathfrak{u}_1/\mathfrak{u}_n,$$

when $k = E_{\mathfrak{p}_2}$.

First we have the following facts about the 2-rank of G (e.g., see [\[Nak79\]](#), which also treats higher-degree fields).

- (i) Suppose $n < 2e + 1$. Then the 2-rank of G is $\lceil \frac{n-1}{2} \rceil f$.
 If 2 is inert in E , then G has 2-rank 0 or 2 when $n = 1$ or $n = 2$, respectively. Hence $G \simeq C_3$ or $G \simeq C_3 \times C_2^2$ in these cases.
 If 2 ramifies in E , then G has 2-rank 0, 1, 1, 2 for $n = 1, 2, 3, 4$, respectively. Necessarily G is isomorphic to C_1 , C_2 , C_4 , or $C_2 \times C_4$.
- (ii) Suppose $n \geq 2e + 1$. Then the 2-rank is $ef + 1 = 3$.
 If 2 is inert in E , this means G has 2-rank 3 when $n \geq 3$. Necessarily $G \simeq C_3 \times C_2^2 \times C_4$ when $n = 3$.
 If 2 ramifies in E , then G has 2-rank 3 when $n \geq 5$. Necessarily $G \simeq C_2^2 \times C_4$ when $n = 5$.

The following more precise information was obtained in [\[Ran10\]](#); these cases respectively correspond to Type 9, Types 6–7, and Type 5 in [\[Ran10\]](#). Write $E = \mathbb{Q}(\sqrt{d})$ where d is squarefree.

- (iii) Suppose 2 is inert in E . Then $G = (\mathfrak{o}_E/2^n\mathfrak{o}_E)^\times \simeq C_3 \times \langle -1 \rangle \times C_{2^{n-1}} \times C_{2^{n-2}}$ for $n \geq 2$. Further, 5 generates a subgroup of index 2 of the third factor, and $3 + 2\sqrt{d}$ generates the last factor. The image of $(\mathbb{Z}/2^n\mathbb{Z})^\times$ in $(\mathfrak{o}_E/\mathfrak{p}_2^n)^\times$ is the subgroup generated by -1 and 5.
- (iv) Suppose $4 \parallel \Delta_E$. The natural map $(\mathbb{Z}/4\mathbb{Z})^\times \rightarrow (\mathfrak{o}_E/\mathfrak{p}^n)^\times$ is injective if and only if $n \geq 3$. Since $(\mathfrak{o}_E/\mathfrak{p}^3)^\times \simeq C_4$, the element $-1 \in \mathfrak{o}_E/\mathfrak{p}^n$ is a square for any $n \geq 3$.

The precise structure of G depends on $d \bmod 8$ and whether $n = 2, 3, 4$ or $n \geq 5$. However, we do not need more than is already stated in (i) and (ii) above, so we simply remark that for any $n \geq 2$ one has $G \simeq C_2 \times C_{2^a} \times C_{2^b}$ for some $a, b \geq 0$ such that $a + b = n - 2$. (If $n \geq 5$, then $|a - b| \leq 1$.)

- (v) Suppose $8 \parallel \Delta_E$. The natural map $(\mathbb{Z}/8\mathbb{Z})^\times \rightarrow G$ is injective if and only if $n \geq 5$. Moreover, $G \simeq \langle -1 \rangle \times C_{2^{r-2}} \times C_{2^s}$, where $s = \lfloor \frac{n}{2} \rfloor$ and $r = \lceil \frac{n}{2} \rceil$ if $n \geq 4$. In this case 5 generates the $C_{2^{r-2}}$ factor, and $1 + \sqrt{d}$ generates the C_{2^s} factor.

We want to know when the image of any nontrivial element of $(\mathbb{Z}/8\mathbb{Z})^\times$ is a square when $n \geq 5$. Necessarily -1 is not a square, and 5 is a square if and only if $n \geq 7$. Further $3 \equiv (-1) \cdot 5$ is never a square.

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