Linear Algebra (MATH 3333) Spring 2009 Section 2
Midterm Practice Problems & Solutions

Throughout the exam, V denotes a vector space.

1. Go over the problems on your quiz and previous homeworks, and make sure you can do them correctly.

True/False

Circle T or F.

2. T F Any two nonzero vectors in $\mathbb{R}^3$ are linearly independent.

False—you need that they are also not scalar multiples of each other (i.e., not collinear) to be L.I.

3. T F A minimal spanning set for V is a basis for V.

True. Being L.I. makes the basis as small as possible.

4. T F Any subspace of $\mathbb{R}^2$ is either a line through the origin or $\mathbb{R}^2$.

False, but almost true. The origin, i.e., the zero subspace $\{0\}$ is also a subspace of $\mathbb{R}^2$.

5. T F The span of two nonzero vectors is either a line through the origin or a plane through the origin.

True. It is a line if they are collinear and a plane if they are L.I.

6. T F The set of polynomials in x of degree at most 5 form a vector space.

True. See the polynomial example P_n from the lecture notes or in Section 4.2 of the text.

Definitions

7. If $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, define span$(S)$.

span(S) is the set of all linear combinations of elements in S, OR
span$(S) = \{a_1v_1 + a_2v_2 + \cdots a_kv_k | a_i \text{ in } \mathbb{R}\}$.

8. With S as above, define what it means for S to be a basis of V.

S is a basis for V if
(a) span$(S) = V$, and
(b) S is linearly independent.

9. With S as above, define what it means for S to be linearly independent.

There are a couple different ways to state this. The simplest is
S is linearly independent if the only solution to

$$a_1v_1 + a_2v_2 + \cdots a_kv_k = 0$$

is the trivial solution:

$$a_1 = a_2 = \cdots = a_k = 0.$$

Another (equivalent) way would be to first define linear dependence, as in the text or lecture.

Problems

Show your work (i.e., prove your answers except where stated otherwise).

10. Show that the set of vectors of the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ in $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$. Find a basis for this space (no proof needed). Describe this space geometrically.

By Theorem 4.3, we only need to show this set, call it V, is closed under addition and scalar multiplication. Let $u = \begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ and $v = \begin{pmatrix} c \\ d \\ c+d \end{pmatrix}$ be *arbitrary* vectors in V. Then $u + v = \begin{pmatrix} a+c \\ b+d \\ (a+b)+(c+d) \end{pmatrix}$ is also in V, i.e., V is closed under addition. Similarly, for any real number k, $ku = \begin{pmatrix} ka \\ kb \\ k(a+b) \end{pmatrix}$ is also in V, so V is closed under scalar multiplication and thus a subspace of $\mathbb{R}^3$.

One basis for V is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Hence V has dimension 2. Geometrically, this means it is a plane in $\mathbb{R}^3$. (If you want to be more precise, you can say it is a plane through the origin, and in fact the plane given by $z = x + y$ in $\mathbb{R}^3$.)

11. Do the same as the previous problem for the subset $\{(x, y, z) : x + y + z = 0\}$ of $\mathbb{R}^3$.

Again, call this set V. Take any $u = (x, y, z)$ and $v = (x', y', z')$ in V. Then we have $x+y+z = x'+y'+z' = 0$. Now their sum $u + v = (x + x', y + y', z + z')$ is also in V because the sum of all the coordinates

$$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0 + 0 = 0.$$

Similarly, if c is any real number then $cu = (cx, cy, cz)$ is in V because

$$cx + cy + cz = c(x + y + z) = c \cdot 0 = 0.$$

This shows V is closed under addition and scalar multiplication, and thus a subspace of $\mathbb{R}^3$.

Since we have one equation in 3 variables, there should be two free variables, i.e., the dimension should be two. For a basis, we can take $\{(1, 0, -1), (0, 1, -1)\}$. This is a plane through the origin in $\mathbb{R}^3$ (obviously the one given by $x + y + z = 0$).

12. Find two different bases for $\mathbb{R}^2$ (no proof needed).

Any two sets of two non-collinear vectors will do, e.g., $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

13. Is $\{(x, y, z) : 2x - 3y + z = 1\}$ a subspace of $\mathbb{R}^3$?

No, for many reasons. The most obvious one is that is does not contain the origin (i.e., the zero vector).

14. Solve the system of equations or conclude that no solutions exist:

$$\begin{aligned} x - y &= 1 \\ 2x - y - z &= 1 \\ -x + 2y - z &= 1. \end{aligned}$$

We reduce the matrix

$$\begin{pmatrix} 1 & -1 & 0 & | & 1 \\ 2 & -1 & -1 & | & 1 \\ -1 & 2 & -1 & | & 1 \end{pmatrix}$$

to get

$$\begin{pmatrix} 1 & -1 & 0 & | & 1 \\ 0 & 1 & -1 & | & -1 \\ 0 & 0 & 0 & | & 3 \end{pmatrix}.$$

But the bottom row means $0 = 3$, a contradiction. So this system is inconsistent, i.e., no solutions exist.

15. Solve the system of equations or conclude that no solutions exist:

$$\begin{aligned} -2x + y + z &= 1 \\ x + z &= 0 \\ x + y - 2z &= -1. \end{aligned}$$

We may write this system as an augmented matrix

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 1 & 1 & -2 & | & -1 \\ -2 & 1 & 1 & | & 1 \end{pmatrix},$$

where we have reordered the equations to make the reduction simpler. This reduces to

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{3} \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & \frac{1}{3} \end{pmatrix}.$$

which means the system has precisely one solution: $(x, y, z) = (-1/3, 0, 1/3)$.

16. Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ linearly independent? If not, find a maximal linear independent subset.

Suppose

$$a \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Reducing the matrix

$$\begin{pmatrix} 1 & 1 & -1 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ -1 & 1 & 1 & | & 0 \end{pmatrix}$$

one gets

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

Hence the only solution is $a = b = c = 0$, i.e., this set is L.I.

17 (#1). Do the same as the previous problem for $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

No. If you put the vectors in a matrix and reduce, you should get a row of all zeros, which will tell you the solutions to the equation

$$a \begin{pmatrix} 1 \\ 2 \\ -2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 1 \\ -2 \\ -2 \\ 3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 0$$

have one free variable.

17 (# 2). Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No. This set has 4 elements, but the dimension of $\mathbb{R}^3$ is 3.

18. Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Yes. Since $\dim \mathbb{R}^3 = 3$, it suffices to check that those three vectors are linearly independent, which is done in #16.

19. Find a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ whose image is the set of vectors of the form $\begin{pmatrix} a \\ 2b - a \\ b \end{pmatrix}$.

Unfortunately, we did not get to this material in lecture so it will not appear as a ordinary exam question, but something along these lines *will* appear on the exam as bonus material. Here is a hint. If $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then the image of A is a subspace V of $\mathbb{R}^n$. If $\{v_1, \ldots, v_m\}$ is a basis for $\mathbb{R}^n$, then $S = \{Av_1, Av_2, \ldots, Av_m\}$ will span the image V of A. (S may or may not be a basis for V.) Think about how you would compute the image of a specific linear transformation A. (This is one of the main reasons we have been studying subspaces, span and basis.)

In case you are still curious, for this problem

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$$

is one possible answer. (Just apply it to an arbitrary vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ to check that it works.)