Linear Algebra (MATH 3333) Spring 2009 Section 2
Final Exam Practice Solutions

Instructions: Try the following on your own, then use the book and notes where you need help. Afterwards, check your solutions with mine online. For Sections 1 and 2, no explanations are necessary. For all other problems, justify your work. There are also bonus problems at the end.

[The bonus questions at the end was a lie. Sorry. There were last time.]

Note: Not every problem on the practice sheet is modeled off of one of your problems for homework. However, you can figure out how to do them with an understanding of the basic concepts from the course. They are designed to help piece together your understanding of the course material. There may be questions

[I don't know what happened to the end of my note. I bet it said something of crucial importance. If I could only remember what.]

1 True/False

In this section A is an $n \times n$ matrix.

1. T F Two vectors are linearly independent if they are not scalar multiples of each other.

True. (Of course linear independence of 3 or more vectors is not so simple.)

2. T F Every square matrix is diagonalizable.

False. We saw $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \neq 0$ is not (no basis of eigenvectors).

3. T F If A is diagonalizable, then there is a basis of eigenvectors of A.

True. In fact the other direction is true also. Theorem 7.3/7.4.

4. T F If λ is an eigenvalue for A, then the eigenspace V_λ is a line.

False. V_λ is always a subspace, and is often a line, but may be a plane or higher dimensional. E.g., for the 2×2 identity matrix, 1 is an eigenvalue with eigenspace $\mathbb{R}^2$.

5. T F If $A = PDP^{-1}$, then $A^3 = P^3D^3P^{-3}$.

False. $A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$.

6. T F $A = P_{T\leftarrow S}[A]_T P_{S\leftarrow T}$ where S is the standard basis for R^n.

False. This is crucial to keep straight for diagonalization: $[A]_T = P_{T\leftarrow S}AP_{S\leftarrow T}$ but $A = P_{S\leftarrow T}[A]_T P_{T\leftarrow S}$. You can keep it straight if you understand the meaning of these equations.

7. T F Any linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ can be composed from (as a matrix, is a product of) rotations, reflections and scalings.

False. Any *isometry* (i.e., a distance preserving map—we talked about these near the beginning of the course) is composed of rotations and reflections (and in fact any such composition is either a rotation or reflection itself). However, the shearing transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not a product of rotations, reflections and scalings. To see that it involves no scaling, observe that 1 is the only eigenvalue. Then it is geometrically clear that the shearing transformation is not composed of rotations and reflections (for instance, it is not an isometry). (Note: to make this argument precise involves a little work, so I would not ask to prove something like this on the final, but I think understanding the ideas of this argument may be helpful.)

8. T F If A is 3×3 with real entries, it cannot have 3 distinct purely imaginary eigenvalues.

True. The characteristic polynomial must be a real cubic polynomial, which must have at least 1 real root (since complex roots occur in conjugate pairs), so it cannot have 3 purely imaginary roots. (If A is allowed complex entries, then the characteristic polynomial may be complex so you can have 3 complex roots.)

2 Short Answer

9. State the definition of a basis for a finite-dimensional vector space V.

10. State the definition of an eigenvalue and an eigenvector for an $n \times n$ matrix A.

See the text or your notes for the definitions.

11. What is the geometric meaning of the λ-eigenspace, V_λ, for A?

V_λ is a subspace (the largest such) on which A just scales all vectors by λ.

12. State three things linear algebra has applications to.

Pretty much any three things you say (which involve math/engineering), you'd be right. For example: Dynamical systems (e.g. population modeling), Markov processes (e.g., weather modeling, Google Pagerank), differential equations (diffusion, sound vibrations, etc.), engineering, computer graphics, economics optimization, GPS modeling, error correcting codes, computer engineering, etc.

13. What is the geometric meaning of a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$?

An action on $\mathbb{R}^3$ which turns any line either (i) into another (or possibly the same) line or (ii) collapses it onto some point. Optional addition: Rotations, reflections, scalings and projections are the basic examples. However more complicated things can happen (as with the shear transformation in 2-dimensions).

14. What is the geometric significance of $\det(A)$ for a 3×3 matrix A?

It (precisely $|\det(A)|$) tells you how much A scales volume by in $\mathbb{R}^3$. Optional addition: Moreover if $\det(A)$ is positive, A is orientation preserving; if $\det(A)$ is negative A is orientation reversing. ($\det(A) = 0$ means A involves a projection.)

15. In studying linear transformations, why does it suffice to study square matrices? Specifically, if $A : \mathbb{R}^2 \to \mathbb{R}^3$ and $B : \mathbb{R}^3 \to \mathbb{R}^2$ are two linear transformations, explain how to study them in terms of square matrices (or equivalently, linear maps from $\mathbb{R}^n \to \mathbb{R}^n$).

Good question. I'm glad you asked. The idea is that we can think a map from $\mathbb{R}^2 \to \mathbb{R}^3$ or a map from $\mathbb{R}^3 \to \mathbb{R}^2$ as a map from $\mathbb{R}^3 \to \mathbb{R}^3$ by identifying $\mathbb{R}^2$ in either case with the xy-plane in $\mathbb{R}^3$.

Specifically for the first case given, $A : \mathbb{R}^2 \to \mathbb{R}^3$ can be thought of as the composition of the standard embedding $(x, y) \to (x, y, 0)$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ composed with *some* map $A' : \mathbb{R}^3 \to \mathbb{R}^3$. In matrix form, this would be

$$A = A' \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where A' is 3×3. (We remark there are lots of A' which will work because we only need to specify what it does on the xy-plane, so what is does to the z-axis can be arbitrary.) This is how things work going from a lower-dimensional space into a higher-dimensional space.

The other case, of a higher-dimensional space into a lower-dimensional space, is similar but a little different. We can think of $B : \mathbb{R}^3 \to \mathbb{R}^2$ as a map $B' : \mathbb{R}^3 \to \mathbb{R}^3$ composed with the projection $(x, y, z) \to (x, y)$, i.e., we can always write

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B'$$

for some 3×3 B' (as before there are many choices for B').

16. What is a Markov process? Give an example. What is the significance of the dominant eigenvectors?

A Markov process is a system with n possible states, such that the probability of being in a certain state at time $t + 1$ is determined by what the state of the system was at time t. See the lecture notes or the text (I think Sec. 8.2) for the rainy day/dry day example.

If A is the Markov matrix and v_0 is a (generic) intial probability vector, then

$$v_\infty := \lim_{t \to \infty} v_t = \lim_{t \to \infty} A^t v_0$$

will be a dominant (probability) eigenvector. Assuming the dominant eigenspace is 1-dimensional, a dominant eigenvector (scale so all entries are positive and sum to 1) tells you the probability of being of being in state i at a random time t (assuming *any* random initial state).

17. Why might you want to exponentiate a matrix?

To obtain a formula for v_t for some (discrete) dynamical system or Markov process (e.g., see # 21.)

18. If A is a 2×2 matrix such that $A\begin{pmatrix}1\\2\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix}$ and $A\begin{pmatrix}3\\-1\end{pmatrix} = 4\begin{pmatrix}3\\-1\end{pmatrix}$, what is A?

There is an easy way to do this and a hard way to do this. The easy way is to realize you have two linearly independent eigenvectors with eigenvalue 4. This means the eigenspace $V_{\lambda=4}$ for A must contain both of these vectors, i.e., $V_{\lambda=4} = \mathbb{R}^2$. Hence A scales all vectors by 4, i.e., $A = 4I = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$. I am scared of the hard way.

3 Problems

19. Let $A = \begin{pmatrix}0 & 0 & -2\\0 & -2 & 0\\-2 & 0 & 3\end{pmatrix}$.
  (i) Find the eigenvectors and eigenvalues of A.
  (ii) Diagonalize A, i.e., write $A = PDP^{-1}$ for some diagonal matrix D.

See p. 465. The only thing that's not done is explicitly writing $A = PDP^{-1}$. Here $P = P_{S \leftarrow T}$ is the matrix of the basis of eigenvectors T, i.e, $P = \begin{pmatrix}0 & -1 & 2\\1 & 0 & 0\\0 & 2 & 1\end{pmatrix}$. Then by row reducing $[P|I]$ we find

$$P^{-1} = \frac{1}{5}\begin{pmatrix}0 & 1 & 0\\-1 & 0 & 2\\2 & 0 & -1\end{pmatrix}.$$

20. (i) Construct a matrix A which rotates $\mathbb{R}^3$ by $\frac{\pi}{4}$ radians counterclockwise around the z-axis.
  (ii) Construct a matrix B which rotates $\mathbb{R}^3$ by $\frac{\pi}{4}$ radians counterclockwise around the line $x = y = z$.

The idea for (i) is the same as for #15, and the idea for (ii) is then to use a change of basis.

(i) Note that a rotation by θ counterclockwise around the z-axis is just

$$\begin{pmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{pmatrix}$$

so

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & \sqrt{2}\end{pmatrix}.$$

(ii) is a little tricky, but the idea is simple. A is rotation around the z-axis, so to rotate around $x = y = z$, we could try to move the line $x = y = z$ to the z-axis, do the rotation given by A, and then move the z-axis back to $x = y = z$. This would then do what B is supposed to.

Let

$$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2} & 0 & 0\\0 & 1 & -1\\0 & 1 & 1\end{pmatrix},$$

which is rotation clockwise by $\frac{\pi}{4}$ around the x-axis. Thus R_1 takes the z-axis to the line given by $z = y$ and $x = 0$. Set

$$R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

which is rotation counterclockwise by $\frac{\pi}{4}$ around the y-axis. Thus R_2 takes the line given by $z = y$ and $x = 0$ to $x = y = z$, so the composition $R_2 \cdot R_1$ takes the z-axis to $x = y = z$. Hence we can write B as

$$B = R_1^{-1} R_2^{-1} A R_2 R_1.$$

I will not multiply this out and you wouldn't have to on an exam either.

This problem is of course too time consuming and a little too tricky (considering you haven't really seen it before) to put on an exam, however the basic idea is I think an important one and you should be able to do a simpler version of this kind of problem (or somewhere in between this and #18) at the time of the final.

As a final remark, you might think that you could write $B = P^{-1}AP$ for *any* change of basis P which takes the z-axis to the line $x = y = z$. However, it is crucial you use an *orthogonal* matrix P (or equivalently, an *isometry*). The point is if you do not do an orthogonal change of basis P when you do the rotation A this will mess up your basis elements before you change back with P^{-1}. These kinds of problems indicate one reason why orthogonal matrices are important.

21. Suppose you have a (discrete) dynamical system given by

$$\begin{aligned} x(t+1) &= x(t) + 2y(t) \\ y(t+1) &= 4x(t) + 3y(t), \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = 1$. Find explicit formulas for $x(t)$ and $y(t)$.

We can rewrite this as:

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

Diagonalize $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. The eigenvalues for A are $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenspaces $\left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} \right\}$. Thus a basis of eigenvectors is $T = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. If S is the standard basis, $P = P_{S \leftarrow T} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and

$$P^{-1} = P_{T \leftarrow S} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 7.3/7.4,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

so

$$A^t = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^t & 0 \\ 0 & 5^t \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5^t + 2(-1)^t & 5^t - (-1)^t \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5^t + (-1)^t \\ 2 \cdot 5^t - (-1)^t \end{pmatrix}.$$

22. Go through Homework 7–12 and make sure you can do each problem, particularly HW 12 since it will not be graded before your final. (Problem A of HW 8 and Problem C of HW 9 are not so important. Most of the others are what I consider to be important material for the course, and you should expect several questions like these on the final.)

I realize you have a lot to review (particularly if you didn't understand things the first time around) and not much time, so try not to get stressed out. Just work hard with what time you have, and do your best on the final. Good luck!