Linear Algebra (MATH 3333) Fall 2007 Sections 1/4
Homework 5 — Practice Problem Solutions

Not to be turned in

Instructions: Try the following on your own, then use the book and notes where you need help. You will be able to check solutions online and bring any questions you may have to Wednesday's class. For Sections 1 and 2, no explanations are necessary. For all other problems, justify your work.

1 True/False

1. T F If A, B, C are 3 $n \times n$ matrices, then $(AB)C = A(BC)$.

True! Associativity of matrix multiplication was Proposition 3 from class, or Theorem 1.2(a) in the book.

2. T F If A is an $m \times n$ matrix, then it defines a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

False! It will give a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

3. T F $\mathbb{R}^n$ is a vector space.

True.

4. T F Any line in $\mathbb{R}^2$ is a linear subspace of $\mathbb{R}^2$.

False! Any line *through the origin* is a linear subspace of $\mathbb{R}^2$, but not any old line.

5. T F In $\mathbb{R}^2$, scaling by k (in all directions) and then rotating by θ clockwise is the same as rotating by θ clockwise and then scaling by k.

Indeed. This is what Proposition 2 says.

2 Short Answer

6. Write down the definition of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

We say T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ if it is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the form

$$T(x, y) = (ax + by, cx + dy)$$

for some real numbers a, b, c, d.

7. State the definition of a linear subspace of $\mathbb{R}^n$.

L is a linear subspace of $\mathbb{R}^n$ if it is the image of some linear transformation into $\mathbb{R}^n$.

8. Geometrically, list three kinds of linear transformations in $\mathbb{R}^2$?

Rotations, reflections and scalings.

9. What is linear algebra?

The study of linear transformations.

10. Geometrically, describe the action of the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$?

We see

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix},$$

so A scales by 2 in the x-direction.

3 Problems

11. Determine the domain, range and image of the linear transformation $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$.

A is a 3×2 matrix, so it maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, i.e. the domain is $\mathbb{R}^2$ and the range is $\mathbb{R}^3$. We compute

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x + 3y \\ -x + y \end{pmatrix}.$$

Hence the image of A is

$$\{(2x, x + 3y, -x + y) | x, y \text{ in } \mathbb{R}\}.$$

12. What does the transformation $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$ do to the line $y = x$, i.e., what is the image of the line $y = x$ under the transformation A?

In vector notation, the line $y = x$ is the line through the origin determined by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the line $y = x$ may be written as the set $\left\{ c \begin{pmatrix} 1 \\ 1 \end{pmatrix} | c \text{ in } \mathbb{R} \right\}$. Note

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}.$$

Thus A maps the line $y = x$ to the line $\left\{ c \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^3$. (You may notice this is the line given by equations $y = 2x, z = 0$ in $\mathbb{R}^3$.)

13. Prove the circle $\{(x, y) | x^2 + y^2 = 1\}$ is not a linear subspace of $\mathbb{R}^2$.

There are several simple proofs. The easiest is that the circle does not contain the origin, but every linear subspace does. The second easiest is that the circle does not contain any lines, so by Corollary 1 cannot be a linear subspace.

14. Let L be the union of the xy-plane and the yz-plane in $\mathbb{R}^3$. Is L a linear subspace of $\mathbb{R}^3$? Justify your answer.

No, by linearity. Specifically, Corollary 3 says that if u and v are in a linear subspace, then so is $u + v$. However $(1, 0, 0)$ and $(0, 0, 1)$ are in L, but their sum $(1, 0, 1)$ is not in L.

15. Go over Homeworks 1–4, and make sure you can do the problems correctly on your own. There will be very similar problems on the exam. (Don't worry about #34 on Homework 3 though.)

Yay! I can do them!