# Linear Algebra (MATH 3333) Fall 2007 Sections 1/4
# Homework 16 — Practice Problem Solutions

**Instructions:** Try the following on your own, then use the book and notes where you need help. You will be able to check solutions online and bring any questions you may have to the last class. For Sections 1 and 2, no explanations are necessary. For all other problems, justify your work.

## 1 True/False

1. T F Two vectors are linearly independent if they are not scalar multiples of each other.

Yes. This was covered on the last exam.

2. T F Every square matrix is diagonalizable.

False. We saw $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is not in the homework.

3. T F If $A$ is diagonalizable, then there is a basis of eigenvectors of $A$.

True. Theorem 7.3/7.4.

4. T F If $\lambda$ is an eigenvalue for $A$, then the eigenspace $V_\lambda$ is a line.

False. $V_\lambda$ is always a subspace, and is often a line, but may be a plane or higher dimensional. E.g., for the $2 \times 2$ identity matrix, 1 is an eigenvalue with eigenspace $\mathbb{R}^2$.

5. T F If $A = PDP^{-1}$, then $A^3 = P^3D^3P^{-3}$.

That's just crazy talk. $A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$.

6. T F $A = P_{T\leftarrow S}[A]_T P_{S\leftarrow T}$.

Wow, I'm just full of falsehoods today. This is crucial to keep straight for diagonalization: $[A]_T = P_{T\leftarrow S}[A]_T P_{S\leftarrow T}$ but $A = P_{S\leftarrow T}[A]_T P_{T\leftarrow S}$. You can keep it straight if you understand the meaning of these equations.

## 2 Short Answer

7. State the definition of a basis for a finite-dimensional vector space $V$.

8. State the definition of an eigenvalue and an eigenvector for an $n \times n$ matrix $A$.

See the text or your notes for the definitions.

9. What is the geometric meaning of the $\lambda$-eigenspace, $V_\lambda$, for $A$?
   $V_\lambda$ is a subspace (the largest such) on which $A$ just scales all vectors by $\lambda$.

10. If $A$ is a $2 \times 2$ matrix such that $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = 4\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, what is $A$?

There is an easy way to do this and a hard way to do this. The easy way is to realize you have two linearly independent eigenvectors with eigenvalue 4. This means the eigenspace $V_{\lambda=4}$ for $A$ must contain both of these vectors, i.e., $V_{\lambda=4} = \mathbb{R}^2$. Hence $V$ scales all vectors by 4, and is $4I = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. I am scared of the hard way.

11. State three things linear algebra has applications to.

Ranking algorithms (e.g. Google), dynamical systems (e.g. population modeling), differential equations (diffusion, sound vibrations, etc.), economics optimization (though we didn't talk about it), GPS modeling, error correcting codes, computer graphics, computer engineering, etc. Linear algebra has is place in pretty much any sort of problem that involves a mathematical model.

# 3 Problems

12. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$.

(i) Find the eigenvectors and eigenvalues of $A$.

(ii) Diagonalize $A$, i.e., write $A = PDP^{-1}$ for some diagonal matrix $D$.

See p. 465. The only thing that's not done is explicitly writing $A = PDP^{-1}$. Here $P = P_{S \leftarrow T}$ is the matrix of the basis of eigenvectors $T$, i.e, $P = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$. Then by row reducing $[P|I]$ we find $P^{-1} = \frac{1}{5}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{pmatrix}$.

13. Suppose you have a (discrete) dynamical system given by

$$\begin{aligned} x(t+1) &= x(t) + 2y(t) \\ y(t+1) &= 4x(t) + 3y(t), \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = 1$. Find explicit formulas for $x(t)$ and $y(t)$.

We can rewrite this as:

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

Diagonalize $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. The eigenvalues for $A$ are $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenspaces $\left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} \right\}$. Thus a basis of eigenvectors is $T = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. If $S$ is the standard basis, $P = P_{S \leftarrow T} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and $P^{-1} = P_{T \leftarrow S} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. By Theorem 7.3/7.4,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & \\ & 5 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

so

$$A^t = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^t & \\ & 5^t \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5^t + 2(-1)^t & 5^t - (-1)^t \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5^t + (-1)^t \\ 2 \cdot 5^t - (-1)^t \end{pmatrix}.$$

14. Go over Homeworks 11–15 and Exam 2, and make sure you can do the problems correctly on your own. There will be similar problems on the final.

15. If you are interested in two drums which sound alike but have different shapes, go here:
*http://www.ams.org/featurecolumn/archive/199706.html*
There is a link to an animation at the bottom.