Linear Algebra (MATH 3333) Fall 2007 Sections 1/4
Homework 14

Due: Mon. Nov. 26, start of class

Instructions: You may **not** use a calculator (or computer). Make sure to write your name, course and section numbers in the top right corner of your solution set, as well as the assignment number on top.

Reading

Sections 7.2

Conceptual Questions

1. What is an eigenspace, and what is the geometric significance of eigenspaces?
2. Why do we want to diagonalize matrices?

Written Assignment

16 points

Problem A. (4 pts) Let A be a 3×3 matrix, i.e., a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. Suppose $T = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$ and

$$[A]_T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Recall, the definition of $[A]_T$ is the 3×3 matrix such that

$$[A]_T[v]_T = [Av]_T$$

for all $v \in \mathbb{R}^3$. Using this definition prove that v_1, v_2, v_3 are eigenvectors for A with respective eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Problem B. (12 pts) Let $A = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix}$.

(i) Find the eigenvalues of A.
(ii) Find the corresponding eigenvectors (or eigenspaces if you prefer) of A.
(iii) Find a basis $T = \{v_1, v_2\}$ for $\mathbb{R}^2$ where v_1 and v_2 are eigenvectors of A.
(iv) Using Theorem 7.3/7.4, write down the matrix $[A]_T$.
(v) If S is the standard basis, determine the transition matrices $P_{S \leftarrow T}$ and $P_{T \leftarrow S}$.
(vi) By Theorem 6.12,

$$[A]_T = P_{T \leftarrow S} A P_{S \leftarrow T}.$$

Compute $[A]_T$ this way, and check it is the same thing you got in (iv).

Bonus Questions

Bonus 1. (a) Let A be a 2×2 *diagonal* matrix, and let $\square$ be the unit square with vertices (0,0), (1,0), (0,1) and (1,1). Prove that A maps $\square$ to a rectangle with area equal to $|\det(A)|$.

(b) Let A be *any* 2×2 matrix, and prove that A maps $\square$ to a parallelogram with area equal to $|\det(A)|$. This means A scales the area of objects by $|\det A|$. (Recall that an earlier homework problem was that reflections and rotations have determinants $+1$ and -1. This means they preserve the area of objects. A negative determinant means A *reverses the orientation* of objects.)

Bonus 2. Using Problem A, prove Theorem 7.3 and Theorem 7.4.