Linear Algebra (MATH 3333) Fall 2007 Sections 1/4
Homework 10 — Practice Problem Solutions

Instructions: Try the following on your own, then use the book and notes where you need help. You will be able to check solutions online and bring any questions you may have to Wednesday's class. For Sections 1 and 2, no explanations are necessary. For all other problems, justify your work.

In the questions below, V denotes a finite-dimensional vector space.

1 True/False

1. T F Any set of linearly independent vectors for V contains a basis for V.

False. Any linearly independent set can be extended to a basis, but most linearly independent sets do not span V.

2. T F Any subspace of $\mathbb{R}^3$ of dimension 2 is a plane through the origin.

True. If W is a subspace of $\mathbb{R}^3$ and $\dim W = 2$, then we can take basis S for W which will consist of two elements, say $S = \{v_1, v_2\}$. Then W is *span* $\{v_1, v_2\}$, which is the (unique) plane through the origin containing the (independent) vectors v_1 and v_2.

3. T F Two vectors are linearly independent if they are scalar multiples of each other.

False. Two vectors are linearly independent if and only if they are *not* scalar multiples of each other.

4. T F Any set of vectors containing the zero vector is linearly dependent.

True. If $S = \{v_1, \ldots, v_k, 0\}$ then the equations

$$a_1 v_1 + \cdots a_k v_k + a_{k+1} 0 = 0$$

has a non-trivial solution, e.g., $a_1 = a_2 = \cdots = a_k = 0$, $a_{k+1} = 1$, i.e., S is linearly dependent.

5. T F P_3 is isomorphic to $\mathbb{R}^3$.

False. The coordinates for $P_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3\}$ are $\left\{ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\} = \mathbb{R}^4$ with respect to the basis $\{1, t, t^2, t^3\}$. Hence P_3 is isomorphic to $\mathbb{R}^4$ but not $\mathbb{R}^3$. Alternatively, P_3 has dimension 4 and $\mathbb{R}^3$ has dimension 3 so they are not isomorphic.

2 Short Answer

6. State the definition of a vector space.

7. State the definition of a basis for V.

8. State the definition of linear independence of a set $v_1, \ldots, v_k$ of vectors in V.

9. State the definition of the dimension of V.

See the text or your notes for the definitions.

10. Determine the rank of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ -1 & -3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{pmatrix}$$

The ranks are 3, 2 and 2, respectively. (Reduce the matrices and count the number of leading 1's, or equivalently, the number of non-zero rows, as in # 11.)

3 Problems

11. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. Determine the image of A in terms of span. What is its dimension?

By Theorem 3, and the observation that Ae_1, Ae_2 and Ae_3 are the columns of A, the image of A is $\text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$. To determine, the dimension, we reduce the matrix A:

$$A \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & -5 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have 3 leading 1's, or 3 nonzero rows, so the columns are linearly independent, i.e., the dimension of the image (the rank) is 3.

12*. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. Determine the kernel of A. What is its dimension?

By the Rank–Nullity Theorem and the last question, the nullity of A is 0, i.e. the kernel of A is the origin $\{0\}$, which has dimension 0. Alternatively, solve the system $Ax = 0$. This question was pretty cheap (by accident), so you may want to try #13 in Section 4.9 for more practice.

13*. Let A be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. If the kernel of A is a line, what is the rank of A?

If the kernel of A is a line, then the nullity of A is 1. By the Rank–Nullity Theorem,

$$rank(A) + nullity(A) = m$$

so $\text{rank}(A) = 1$.

14. Go over Homeworks 6–9, and make sure you can do the problems correctly on your own. There will be similar problems on the exam.

Done.

Bonus. Prove any subspace of $\mathbb{R}^n$ (a vector space) is a linear subspace of $\mathbb{R}^n$ (the image of a linear transformation).

Wait till after the test!