Linear Algebra (MATH 3333) Fall 2007 Section 4
Exam 2 — Solutions

Instructions: Please write your name at the top. Calculators, notes, books, etc. are not allowed. For Sections 1 and 2, you need not show any work. For Section 3, explain your solutions. You may use the back of the sheets as scrap paper. Also, if you run out of room on the front, you may continue your work on the back provided you note this on your exam.

1 True or False? You be the judge.

Each question is worth 1 point. Circle T or F.

1. T F Two vectors are linearly independent if and only if one is not a scalar multiple of the other.

True. $a_1v_1 + a_2v_2 = 0 \iff a_1v_1 = (-a_2)v_2$ has a nontrivial solution if and only if v_1 and v_2 are not scalar multiples.

2. T F Any subspace of $\mathbb{R}^3$ of dimension 2 is a plane through the origin.

True. Pick a basis v_1, v_2 for this subspace. Then it is $\text{span}\{v_1, v_2\}$ = plane through v_1 and v_2.

3. T F $\text{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$ is a line.

True. Even though there are two vectors here, they are scalar multiples.

4. T F Any subset of a linearly dependent set is linearly dependent.

False. The subset $\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}$ of the linearly dependent set in #3 is linearly independent. In fact, for any linearly dependent set, we can always throw away some vectors to get something linearly independent.

5. T F Any two bases for a vector space have the same number of elements.

True. This number is the dimension.

2 Short Answers

Each question is worth 1 point. You need not show your work.

6. Let V be a vector space. State the definition of a basis of V.

A subset S of V is a basis for V if
(a) $\text{span}(S) = V$, and
(b) S is linearly independent.

7. Let V be a vector space. State the definition of the dimension of V.

The number of elements in a basis for V.

8. Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V. State what it means for $v_1, v_2, \ldots, v_n$ to be linearly independent.

The only solution to

$$a_1v_1 + a_2v_2 + \cdots a_nv_n = 0$$

is the trivial one

$$a_1 = a_2 = \cdots = a_n = 0.$$

9. Find a basis for $\mathbb{R}^3$ which contains the vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Lots of things will work, you just need to pick any v_2, v_3 such that v_1, v_2, v_3 are linearly independent. E.g.,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is a basis for $\mathbb{R}^3$.

10. Consider $S = \{t^2 + 2, t - 1, t + 1\}$ as a basis for $P_2 = \{a_0 + a_1 t + a_2 t^2 | a_i \text{ real}\}$. Suppose $[v]_S = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

What is v?

Recall $[v]_S = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ means

$$v = 0(t^2 + 2) - 1(t - 1) + (t + 1) = 2.$$

11. Let $v = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$, and consider the basis $S = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$. What is $[v]_S$?

Write

$$v = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(You can either figure this out in your head, or solve the linear system

$$\begin{pmatrix} 1 & 1 & | & 4 \\ -1 & 1 & | & 0 \end{pmatrix}.)$$

Thus

$$[v]_S = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

12. Is $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a + 2b - c = 1 \right\}$ is a subspace of $\mathbb{R}^3$?

No, for lots of reasons. It doesn't contain zero, is not closed under addition, and is not closed under subtraction.

3 Problems

Show your work.

13. (8 pts) Let $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$. Determine

(a) image(A),
(b) rank(A),
(c) ker(A),
(d) nullity(A).

(a) The easiest answer is: image $(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Our computations in (b) will show that we can just take the span of the first three vectors, i.e., a better answer is image $(A) = \mathbb{R}^3$.

(b), (d) One needs to row reduce A. I won't write out the steps, but you should get something like

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

There are three columns with leading 1's, and one without, so we get

$$\text{rank}(A) = 3,$$

$$\text{nullity}(A) = 1,$$

(c) The kernel is the set of solutions $\{v\}$ to $Av = 0$, i.e., the solutions to the (row-reduced) system

$$\begin{pmatrix} 1 & 1 & -1 & 1 & | & 0 \\ 0 & 1 & -1 & 0 & | & 0 \\ 0 & 0 & 1 & 1 & | & 0 \end{pmatrix}.$$

Hence

$$\ker(A) = \left\{ \begin{pmatrix} -a_4 \\ -a_4 \\ -a_4 \\ a_4 \end{pmatrix} \right\},$$

i.e., the line in $\mathbb{R}^4$ through $\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ (or $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ if you don't like minus signs).

14. (2 pts) Solve

$$\begin{aligned} 2x + 3y &= 2 \\ 5x - 4y &= 9. \end{aligned}$$

I.e., solve

$$\begin{pmatrix} 2 & 3 & | & 2 \\ 5 & -4 & | & 9 \end{pmatrix}.$$

Replacing the second row with twice the second row minus 5 times the first row gives

$$\begin{pmatrix} 2 & 3 & | & 2 \\ 0 & -23 & | & 8 \end{pmatrix}.$$

Hence

$$y = -\frac{8}{23}$$

and

$$x = 1 - \frac{3}{2}y = 1 + \frac{12}{13} = \frac{35}{23}.$$

15. (2 pts) Let A be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ and suppose the image of A is a plane. What is the dimension of the kernel of A? Prove your answer.

The dimension of the image of A is called the rank of A and the dimension of the kernel of A is called the nullity of A. The Rank-Nullity Theorem asserts rank + nullity = dimesion of the domain, i.e.,

$$\text{rank}(A) + \text{nullity}(A) = 4.$$

If the image of A is a plane, then $\text{rank}(A) = 2$ so $\dim(\ker(A)) = \text{nullity}(A) = 2$.

Bonus 1. Prove any subspace of $\mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$.

Let V be a subspace of $\mathbb{R}^n$. Then $V = \text{span}(S)$ for some subset $S = \{v_1, v_2, \cdots, v_k\}$ of V. Let A be the $(n \times k)$ matrix whose i–th column is v_i. Then

$$\text{image}(A) = \text{span}\{v_1, v_2, \cdots, v_k\} = V.$$

The definition of linear subspace of $\mathbb{R}^n$ was the image of some linear transformation, and we have just constructed a linear transformation whose image is V, i.e., V is a linear subspace of $\mathbb{R}^n$.

Bonus 2. Let A be an 3×3 matrix. The *row space* of A is the span of the rows of A. The *row rank* of A is the dimension of the row space. Prove the row rank of A equals the (column) rank of A.

See p. 275.